Input-to-State Stabilization with Quantized Output Feedback*

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Abstract. We study control systems where the output subspace is covered by a finite set of quantization regions, and the only information available to a controller is which of the quantization regions currently contains the system's output. We assume the dimension of the output subspace is strictly less than the dimension of the state space. The number of quantization regions can be as small as 3 per dimension of the output subspace. We show how to design a controller that stabilizes such a system, and makes the system robust to an external unknown disturbance in the sense that the closed-loop system has the Input-to-State Stability property. No information about the disturbance is required to design the controller. Achieving the ISS property for continuous-time systems with quantized measurements requires a hybrid approach, and indeed our controller consists of a dynamic, discrete-time observer, a continuous-time state-feedback stabilizer, and a switching logic that switches between several modes of operation. Except for some properties that the observer and the stabilizer must possess, our approach is general and not restricted to a specific observer or stabilizer. Examples of specific observers that possess these properties are included.

1 Introduction

Many tools developed in control theory assume a system where the measurement that enters the controller is either the state of the system (state-feedback) or some linear transformation of the state (output-feedback). In many practical applications, however, the measurement available to the controller is only a quantized version of the aforementioned signals. More specifically, the measurement available to the controller is confined to a finite set of values. While the size of this finite set of values is assumed to be fixed, we do assume that the mapping from the output-subspace into this set depends on a few parameters that can be changed by the controller. This is referred to as dynamic quantization. Quantization can result from the physical properties of the sensors in the system. For example a coarse temperature sensor which can only measure "normal", "too hot", or "too low", but its threshold can be adjusted. Another example is a low

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resolution camera whose orientation and optical zoom can be adjusted. Quantization can also result from a link with a limited data rate between the sensors and the controller. The approach in this paper is especially designed for systems where each sensor is connected, through some limited data rate link, directly to the controller. In particular this means there is no need for a processing unit on the “sensor side” to collect the information from all the sensors, and generate a state estimate from the partial output measurements, before transmitting it to the controller. Basic references on quantized control include [1], [2] and [3].

Several different notions of stability exist in the literature. We chose the notion of Input-to-State Stability (ISS), first presented in [4] for continuous-time systems. Roughly speaking, a system is ISS if every state trajectory corresponding to a bounded disturbance remains bounded, and the trajectory eventually becomes small if the disturbance is small (no matter what the initial state is). The notion of ISS was extended to discrete-time systems in [5]. Our choice of ISS as the desired property is natural because we want to have a bounded response to arbitrary bounded disturbances. This implies, in particular, that no information about the disturbance bound is given to the controller.

Recent papers on how to achieve stabilization under quantization include: [6], [7], [8] and [9] which assume only disturbance-free systems; [10] and [11] which deal only with disturbances whose bound is known to the controller; [12] which only requires the controller to know some statistical information about the disturbance but not its bound; and [13] and [14] in which the controller does not have any information about the disturbance. Even though in [12] and [13] the controller does not know the disturbance bound, neither shows ISS — [12] shows mean square stability in the stochastic setting and [13] shows stability in probability. The paper [14] does show ISS; however, the approach in [14] is considerably different from our approach and in particular it does not guarantee a minimum number of quantization regions or a minimum data rate. Of the papers that deal with disturbances, only [12] and [13] also deal with the output-feedback case. However, in contrast to our paper, in these papers it is assumed that the quantization is applied after a state estimate is constructed by some observer that has direct access to the measurements. This approach is arguably less relevant in applications since it does not address the case where the quantization is due to physical or practical limitations of the sensors (and not only due to a limited data rate).

The work presented in this paper is built on our recent work [15], which was the first to show how to achieve ISS under state-quantization and minimum data rate. In the work presented here, we show how to extend that scheme to output-feedback systems where only the projection of the state into a lower dimensional subspace is measured (and then quantized). Achieving the ISS property for continuous-time systems with quantized measurements requires a hybrid approach, and indeed our controller consists of a dynamic, discrete-time observer, a continuous-time state-feedback stabilizer, and a switching logic that switches between several modes of operation.

The paper is organized as follows. In §2 we define the system and the quantizer. In §3 we give an overview of and the motivation for the three modes of operation of our controller. In §4 we define a general form of an observer, and then present the controller that achieves the objectives listed above. Our main result is presented in that section, and it is followed by a simulation. In §5 we give examples of specific observers that can be used with our control system. We conclude in §6. Due to paper length limitations we are unable to include here the proof of our result. It will appear in the journal version of this work, while in the meantime it can be viewed in the appendix of the review version of this paper, which is available at `http://decision.csl.uiuc.edu/~ysharon/hscc08_full.pdf`

2 System Definition

The continuous-time dynamical system we are to stabilize is as follows ($t \in \mathbb{R}_{\geq 0}$, $k \in \mathbb{N} \cup \{0\}$):

$$\begin{aligned} &\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t) + D\boldsymbol{w}(t) \\ &\boldsymbol{y}(k) = C\boldsymbol{x}(kT_s) \qquad \boldsymbol{z}(k) = Q\left(\boldsymbol{y}(k); \boldsymbol{c}(k), \mu(k)\right) \end{aligned} \tag{1}$$

where $\boldsymbol{x} \in \mathbb{R}^{n_x}$ is the state of the system, $\boldsymbol{u} \in \mathbb{R}^{n_u}$ is the control input that the control system will need to generate, $\boldsymbol{w} \in \mathbb{R}^{n_w}$ is an unknown disturbance which is injected to the system and $\boldsymbol{y} \in \mathbb{R}^{n_y}$ is the projection of the state space into the output subspace which is measured by the sensors. Finally, $\boldsymbol{z} \in \mathbb{R}^{n_y}$ is the information available to the controller. We use T_s for the time interval between subsequent measurement. We will refer to each instance in time when a measurement is taken as a time sample. A, B, D, C are real matrices of appropriate dimensions. We assume that A and B are a controllable pair and that A and C are an observable pair.

We use N for the number of quantization regions per observed dimension. It can be determined by the physical properties of the sensor or from the data rate. Given N, the data rate required, in bits per time sample, is $R = \log_2\left(N^{n_y}\right)$. Our quantizer, denoted by Q, is parameterized by $\boldsymbol{c} \in \mathbb{R}^{n_y}$ and $\mu \in \mathbb{R}$ as follows (see Figure 1 for an illustration):

$$Q_i\left(\boldsymbol{x}; \boldsymbol{c}, \mu\right) \doteq c_i + \begin{cases} (-N+1)\mu & x_i - c_i \leq (-N+2)\mu \\ (-N+3)\mu & (-N+2)\mu < x_i - c_i \leq (-N+4)\mu \\ \vdots & \vdots \\ 0 & -\mu < x_i - c_i \leq \mu \\ \vdots & \vdots \\ (N-3)\mu & (N-4)\mu < x_i - c_i \leq (N-2)\mu \\ (N-1)\mu & (N-2)\mu < x_i - c_i . \end{cases} \tag{2}$$

We will refer to $\boldsymbol{c}$ as the center of the quantizer, and to μ as zoom factor. Note that what will actually be transferred from the quantizer to the observer will be an index to one of the quantization regions. The observer, which knows the values of $\boldsymbol{c}$ and μ, will use this information to convert the received index to the

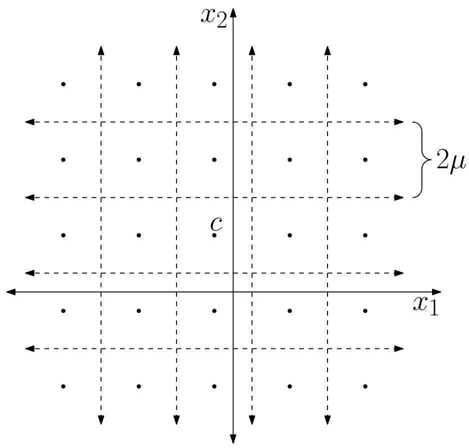

Fig. 1. Illustration of the quantizer for the two-dimensional output subspace, $N = 5$. The dashed lines define the boundaries of the quantization regions. The black dots define where the quantizer estimates the projection of the state to be, given the index of the quantization region that currently contains the projection.

value of Q as given in (2). The controller sets $\boldsymbol{u}$, $\boldsymbol{c}$ and μ, and the only signal directly observed by the controller is $\boldsymbol{z}$. The system model, represented by the matrices A, B, D, C, is known to the controller.

Adopting the standard ISS notion to the class of hybrid systems that we design here, we will say that the closed-loop system is ISS if its solution satisfies

$$|\boldsymbol{x}(t)| \leq \beta_{cl}\left(|\boldsymbol{x}(0)|, t\right) + \gamma_{cl}\left(\|\boldsymbol{w}\|\right), \qquad \forall t \geq 0 \tag{3}$$

for some $\mathcal{K}_\infty$-function[1] γ_{cl} and some $\mathcal{KL}$-function[2] β_{cl}. See also [16] for a study of ISS in the framework of impulsive systems.

In this paper we will use the ∞-norm unless otherwise specified: for vectors, $|\boldsymbol{x}| \doteq |\boldsymbol{x}|_\infty \doteq \max_i |x_i|$; for matrices, $\|M\| \doteq \max_{\boldsymbol{x}} \frac{|M\boldsymbol{x}|}{|\boldsymbol{x}|} \equiv \max_i \left(\sum_j |M_{ij}|\right)$; for continuous-time signals, $\|\boldsymbol{w}\|_{[t_1,t_2]} \doteq \max_{t\in[t_1,t_2]} |\boldsymbol{w}(t)|_\infty$, $\|\boldsymbol{w}\| \doteq \|\boldsymbol{w}\|_{[0,\infty)}$; and for discrete-time signals, $\|\boldsymbol{y}\|_{\{k_1\ldots k_2\}} \doteq \max_{k\in\{k_1\ldots k_2\}} |\boldsymbol{y}(k)|_\infty$, $\|\boldsymbol{y}\| \doteq \|\boldsymbol{y}\|_{\{0\ldots\infty\}}$.

3 Overview of the Controller Design

Our controller operates in three different modes of operation. The motivation for each of these modes is given in this section.

A general quantizer may consist of quantization regions of finite size, for which the estimation error can be bounded, and regions of infinite size, where the

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[1] A function $\alpha : [0,\infty) \to [0,\infty)$ is said to be of class $\mathcal{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$. A function $\alpha : [0,\infty) \to [0,\infty)$ is said to be of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and also unbounded

[2] A function $\beta : [0,\infty) \times [0,\infty) \to [0,\infty)$ is said to be of class $\mathcal{KL}$ if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each fixed $t \geq 0$ and $\beta(s,t)$ decreases to 0 as $t \to \infty$ for each fixed $s \geq 0$.

estimation error can not be bounded. We will refer to these regions as bounded and unbounded regions, respectively. Due to the fact that there are only a finite number of quantization regions to cover the infinite-size $\mathbb{R}^{n_y}$ output subspace, only a region of finite size of this subspace can be covered by the bounded regions. The size of this region, however, can be adjusted dynamically by changing the parameters of the quantizer. We refer to this region which is covered by only bounded quantization regions as the unsaturated region. Our controller follows the general framework which was introduced in several previous papers to achieve disturbance rejection using dynamic quantization. This framework consists of two main modes of operation, generally referred to as a "*zoom-in*" and a "*zoom-out*" mode. During the *zoom-out* mode the unsaturated region is enlarged until the measured output is captured in this region and a bound on the estimation error can be established. This is followed by a switch to the *zoom-in* mode. During the *zoom-in* mode the size of the quantization regions is reduced in order to have the state estimate converge to the true state. The reduction of the size of the quantization regions inevitably reduces the size of the unsaturated region. As the size of this region is reduced, eventually the unknown disturbance will drive the measured output outside the unsaturated region. To regain a bounded estimation error, the controller will switch back to the *zoom-out* mode. By switching repeatedly between these two modes, ISS relation can be established. In this paper we use the name "*capture*" mode for the *zoom-out* mode

In our quantizer there are $2n_y$ unbounded quantization regions. If each sensor measures only one dimension of the output subspace, then this setting allows for an independent reading by each sensor. This setting also allows the use of as little as 3 quantization regions per dimension. To achieve the minimum data-rate, however, we are required to use the unbounded regions not only to detect saturation (as is done in previous papers), but also to reduce the estimation error. Consider for example the case of only 3 quantization regions for each dimension. In this case there is only one bounded region which can not be used by itself to reduce the estimation error. This dual use is done by dividing the *zoom-in* mode into two modes: a "*measurement-update*" mode and an "*escape-detection*" mode. After receiving r successive measurements in bounded quantization regions, where r is the observability index, and assuming there are no disturbances, we are able to define a containment region in the state space which must contain the state. We enlarge this region by a constant to accommodate some disturbance. In the *measurement-update* mode we cover this containment region using both the bounded and the unbounded regions of the quantizer. This way we are able to use the smallest quantization regions, which leads to the fastest reduction in the estimation error. The "problem" with this mode is that if a strong disturbance comes in, we will not be able to detect it. Therefore, in the *escape-detection* mode we use larger quantization regions, but cover the containment region using only the bounded regions. Thus, if a strong disturbance does come in, we will be able to detect it as it will drive the measured output to one of the unbounded regions.

The precise details on how to design to controller are given in the next section.

4 Controller Design

We define the sampled-time versions of A, $\boldsymbol{u}$ and $\boldsymbol{w}$ as:

$$A_d \doteq \exp\left(T_s A\right), \qquad \boldsymbol{u}_d\left(k\right) \doteq \int_0^{T_s} \exp\left(A\left(T_s - t\right)\right) B\boldsymbol{u}\left(kT_s + t\right) dt,$$

$$\boldsymbol{w}_d\left(k\right) \doteq \int_0^{T_s} \exp\left(A\left(T_s - t\right)\right) D\boldsymbol{w}\left(kT_s + t\right) dt,$$

so we can write $\boldsymbol{x}\left(\left(k+1\right)T_s\right) = A_d\boldsymbol{x}\left(kT_s\right) + \boldsymbol{u}_d(k) + \boldsymbol{w}_d(k)$. We assumed that A and B are a controllable pair, so there exists a control gain K such that $A+BK$ is Hurwitz. By construction A_d is full rank, and in general (unless T_s belongs to some set of measure zero) the observability of A and C implies that A_d and C are an observable pair. Thus there exists r such that:

$$\widetilde{C} \doteq \begin{pmatrix} CA_d^{-r+1} \\ \vdots \\ CA_d^{-1} \\ C \end{pmatrix} = \begin{pmatrix} C \\ CA_d \\ \vdots \\ CA_d^{r-1} \end{pmatrix} A_d^{-r+1} \tag{4}$$

has full column rank.

Our controller consists of three elements: an observer which generates a state estimate; a switching logic which sets the parameters for the quantizer and for the observer; and a stabilizing control law which computes the control input based on a state estimate. For simplicity of presentation, we assume the stabilizing control law is a simple static gain given by K. However, any control law that will render the closed-loop system ISS with respect to the disturbance and the estimation error, will work with our controller. Note that it is sufficient for K to be such that $A + BK$ is Hurwitz in order to satisfy this ISS requirement. In the next subsection we present a general structure for an observer, and specify the properties it is required to satisfy. In subsection 4.2 we present the algorithm for the switching logic and state our main theorem.

4.1 Desired Observer Properties

The first element in our control system is the observer. The observer is required to generate an estimate of the state based on current and previous quantized measurements. We assume that the observer is linear, and that there exists a sequence of linear gains, G_0, G_1, …, G_{d-1}, $d > r$, where $G_k \in \mathbb{R}^{n_x \times (k+r)n_y}$, such that the state estimate can be written for $k \in \{0 \ldots d-1\}$ as:

$$\hat{\boldsymbol{x}}_u(k_0 + k) = G_k \begin{bmatrix} \boldsymbol{z}(k_0 - r + 1) + C\sum_{i=1}^{k+r-1} A_d^{-i}\boldsymbol{u}_d(k_0 - r + i) \\ \vdots \\ \boldsymbol{z}(k_0 + k - 1) + CA_d^{-1}\boldsymbol{u}_d(k_0 + k - 1) \\ \boldsymbol{z}(k_0 + k) \end{bmatrix}.$$

Note that we must have at least r successive measurements to generate a state estimate. Therefore, (5) is defined only for $k_0 \geq r - 1$. We use the subscript

u to indicate that $\hat{\boldsymbol{x}}_u(k)$ is our estimate of $\boldsymbol{x}(k)$ based on measurements up to $\boldsymbol{z}(k)$. We will later also use the subscript p to indicate that $\hat{\boldsymbol{x}}_p(k)$ is our estimate of $\boldsymbol{x}(k)$ based on measurements up to $\boldsymbol{z}(k-1)$. The subscripts u and p stand for *update* and *predict*, respectively, which are common notations in the Kalman filter. We denote the quantization error by $\boldsymbol{e}_q(k) \doteq \boldsymbol{z}(k) - \boldsymbol{y}(k)$ and the state estimation error by $\boldsymbol{e}_x(k) \doteq \hat{\boldsymbol{x}}_u(k) - \boldsymbol{x}(kT_s)$.

The first requirement for our approach to succeed is that the linear gains G_0, G_1, …, G_d are such that if no disturbance is injected into the system, and $\boldsymbol{e}_q \equiv 0$, then the state estimate is exact: $\boldsymbol{e}_x \equiv 0$. In the presence of estimation errors, $\boldsymbol{e}_q \neq 0$, and bounded disturbances, the state estimate cannot be exact, but we will need it to converge to the true state. This is achieved by having $\mu(k+d) < \mu(k)$. We cannot, however, decrease μ arbitrarily, since we need the quantization regions to cover the projection of the containment region into the output subspace. The containment region is the region where we expect the state to be based on previous measurements, and given that the disturbances are small enough. If at time sample $k-1$ the gain G_p was used then at time sample k the radius (in ∞-norm) of the projection of this containment region is given by $F(\mu, k, p) + \alpha \|\mu\|_{\{k-p-r \dots k-p-1\}}$. F (see (6) below for a precise definiton) is the radius if there are no disturbances, and α is used as a "slack" for the disturbance. Note that the only variable on which this radius depends is μ, and the dependence is linear. Thus we can arbitrarily choose the initial values of μ in order to verify if we get convergence. The requirement is thus formulated as follows: there exist $\alpha \in \mathbb{R}_{>0}$ and $\sigma < 1$ ($\sigma \in \mathbb{R}_{>0}$) such that if we set

$$\begin{aligned} \mu'(k) &= 1, && k \in \{0 \dots r-1\} \\ \mu'(k) &= \frac{F(\mu'; k; k-r) + \alpha}{N}, && k \in \{r \dots d-1\} \\ \mu'(k) &= \frac{F(\mu'; k; k-r) + \alpha}{N-2}, && k \in \{d \dots d+r-1\}, \end{aligned} \tag{5}$$

where

$$F(\mu; k; p) \doteq \max_{i \in \{1 \dots n_x\}} \sum_{l=-r}^{p-1} \sum_{m=1}^{n_y} \left| (CA_d G_p)_{i,(l+r)n_y+m} \right| \mu(k-p+l). \tag{6}$$

then

$$\|\mu'\|_{k \in \{d \dots d+r-1\}} \leq \sigma. \tag{7}$$

The first line in (5) corresponds to our arbitrary choice of initial values for μ. The second and third lines correspond to the minimal possible value for μ in the *measurement-update* mode and in the *escape-detection* mode, respectively. If the observer satisfies this second requirement for some α, we say that it has the *convergence property* for this α. Note that if it has this property for some α_0 then it will have it for all $\alpha < \alpha_0$. Note also that it is possible to satisfy this requirement just by increasing N sufficiently.

4.2 Switching Logic

The controller will operate in one of three modes which will be determined by the switching logic: *capture*, *measurement update* or *escape detection*. The initial mode will be *capture*. The current mode will be stored in the variable $mode(k) \in \{capture, update, detect\}$. The controller will also use[3] $\hat{\boldsymbol{x}}_p(k) \in \mathbb{R}^{n_x}$, $\hat{\boldsymbol{x}}_u(k) \in \mathbb{R}^{n_x}$, $\hat{\boldsymbol{x}}(t) \in \mathbb{R}^{n_x}$, $p(k) \in \mathbb{Z}$ and $saturated \in \{true, false\}$ as auxiliary variables. We initialize $\hat{\boldsymbol{x}}_p(0) = \mathbf{0}$. The initial value of $\mu(0)$, the zoom factor for the quantizer, can be any positive value and it will be regarded as a design parameter. The controller will also have three other design parameters: $\alpha \in \mathbb{R}_{>0}$, $s \in \mathbb{R}_{>0}$, and $\Omega_{\text{out}} \in \mathbb{R}$, $\Omega_{\text{out}} > \|A\|$. With a slight abuse of notation we define:

$$G(\boldsymbol{z}; \boldsymbol{u}_d; k; p) \doteq G_p \begin{bmatrix} \boldsymbol{z}(k-r-p+1) + C\sum_{i=1}^{p+r-1} A_d^{-i}\boldsymbol{u}_d(k-r-p+i) \\ \vdots \\ \boldsymbol{z}(k-1) + CA_d^{-1}\boldsymbol{u}_d(k-1) \\ \boldsymbol{z}(k) \end{bmatrix}.$$

At each time sample, k, the following switching logic will be executed:

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1 Preliminaries

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if $mode(k) = capture$ **then**
  set $\mu_k = \Omega_{out}\mu_{k-1}$
else if $mode(k) = update$ **then**
  set
$$\mu_k = \frac{F(\mu; k; p(k-1)) + \alpha\|\mu\|_{\{k-r-p(k-1)\ldots k-1-p(k-1)\}}}{N} \tag{8}$$
else if $mode(k) = detect$ **then**
  set
$$\mu_k = \frac{F(\mu; k; p(k-1)) + \alpha\|\mu\|_{\{k-r-p(k-1)\ldots k-1-p(k-1)\}}}{N-2} \tag{9}$$
end if
have the observer record $\boldsymbol{z}(k) = Q(\boldsymbol{y}(k); C\hat{\boldsymbol{x}}_p(k), \mu_k)$
if $\exists i$ such that $\boldsymbol{z}_i(k) = (C\hat{\boldsymbol{x}}_p(k))_i \pm (N-1)\mu_k$ **then**
  set $saturated(k) =$ **true**
else
  set $saturated(k) =$ **false**
end if
by default the mode will not change – set $mode(k+1) = mode(k)$

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[3] The distinction between $\hat{\boldsymbol{x}}_u$, $\hat{\boldsymbol{x}}_p$ and $\hat{\boldsymbol{x}}$ is only to make the proofs easier to read. The controller can be implemented using just one variable.

2 *capture* mode

if $mode(k) = capture$ **then**
  if $saturated(k)$ **then**
    set $p(k) = 0$ and use the observer to update $\hat{\boldsymbol{x}}_u(k) = \hat{\boldsymbol{x}}_p(k)$
  else
    set $p(k) = p(k-1) + 1$
    if $p(k) = r$ **then**
      set $p(k) = 0$ and use the observer to compute $\hat{\boldsymbol{x}}_u(k) = G(\boldsymbol{z}; \boldsymbol{u}_d; k; 0)$
      switch to the *measurement update* mode: set $mode(k+1) = update$
    else
      use the observer to update $\hat{\boldsymbol{x}}_u(k) = \hat{\boldsymbol{x}}_p(k)$
    end if
  end if
end if

3 *measurement update* mode

if $mode(k) = update$ **then**
  set $p(k) = p(k-1)+1$ and use the observer to compute $\hat{\boldsymbol{x}}_u(k) = G(\boldsymbol{z}; \boldsymbol{u}_d; k; p(k))$
  if $p(k) = d - r$ **then**
    switch to the *escape detection* mode: set $mode(k+1) = detect$
  end if
end if

4 *escape detection* mode

if $mode(k) = detect$ **then**
  if not $saturated(k)$ **then**
    set $p(k) = p(k-1) + 1$
    if $p(k) < d$ **then**
      use the observer to compute $\hat{\boldsymbol{x}}_u(k) = G(\boldsymbol{z}; \boldsymbol{u}_d; k; p(k))$
    else
      set $p(k) = 0$ and use the observer to compute $\hat{\boldsymbol{x}}_u(k) = G(\boldsymbol{z}; \boldsymbol{u}_d; k; 0)$
      switch to the *measurement update* mode: set $mode(k+1) = update$
    end if
  else
    set $p(k) = 0$, $\mu(k) = s$ and use the observer to update $\hat{\boldsymbol{x}}_u(k) = \hat{\boldsymbol{x}}_p(k)$
    switch to *capture* mode: set $mode(k+1) = capture$
  end if
end if

Between the time samples the following will be executed:

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5 Control input generation

use the observer to update $\hat{\boldsymbol{x}}(kT_s) = \hat{\boldsymbol{x}}_u(k)$; $\boldsymbol{u}_d(k) = 0$
for $t \in [0, T_s)$ **do**
  use the stabilizing control law to set the control action $\boldsymbol{u}(kT_s + t) = K\hat{\boldsymbol{x}}(kT_s + t)$
  use the observer to update:

$$\begin{aligned} \dot{\hat{\boldsymbol{x}}}(kT_s + t) &= A\hat{\boldsymbol{x}}(kT_s + t) + B\boldsymbol{u}\,(kT_s + t) \\ \dot{\boldsymbol{u}}_d(k) &= \exp\left(A\left(T_s - t\right)\right) B\boldsymbol{u}\,(kT_s + t) \end{aligned}$$

end for
use the observer to update $\hat{\boldsymbol{x}}_p\,(k+1) = \lim_{t \nearrow T_s} \hat{\boldsymbol{x}}\,(kT_s + t)$

We are now ready to state our main result (see the last paragraph in §1 for a reference to the proof):

Theorem 1 *Consider the system* (1)*. If we implement the controller with the algorithm above, and the observer has the convergence property for the α chosen for the implementation, then the closed-loop system will be input-to-state stable with respect to the disturbances.*

An illustrative simulation of this controller is given in figure 2.

5 Observer Examples

In §4.1 we gave a somewhat cumbersome definition for an observer. The reason was to allow for our approach to be implemented with a wide range of observers. In this section we give two examples of observers for which our definition is valid.

5.1 Pseudo-Inverse Observer

Perhaps the most obvious observer is the pseudo-inverse observer[4]:

$$G_0 = \left(\widetilde{C}^T\widetilde{C}\right)^{-1}\widetilde{C}^T, \qquad G_i = \left[\,0_{n_x \times n_y} \mid G_{i-1}\,\right], \quad \forall i \in \{1 \ldots d-1\} \tag{10}$$

where $\widetilde{C}$ is defined in (4). Since our assumption is that $\widetilde{C}$ has full column rank, then $G_0\widetilde{C} = I$, the identity matrix. Thus if no disturbance is injected into the system, and there is no quantization error, then indeed the state estimate will be exact. This satisfies the first requirement from §4.1. A sufficient condition for this observer to satisfy the second requirement from §4.1 is

$$\sigma_{pi} \doteq \frac{1}{N}\left\|CA_dG_0\right\| < 1. \tag{11}$$

[4] $0_{n_x \times n_y}$ is the zero matrix of dimension $n_x \times n_y$

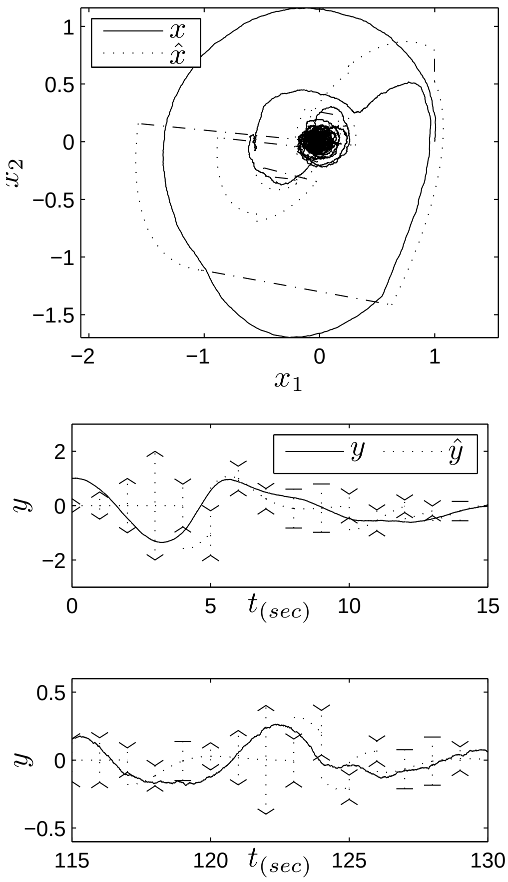

Fig. 2. Simulation of the proposed controller. Simulated here is a two dimensional dynamical system: $\dot{\boldsymbol{x}}(t) = [0.1, -1; 1, 0.1]\,\boldsymbol{x}(t) + [0; 1]\,\boldsymbol{u}(t) + [1, 0; 0, 1]\,\boldsymbol{w}(t)$, where only the first dimension is observed, $\boldsymbol{y}(k) = [0, 1]\,\boldsymbol{x}\,(kT_s)$, through a quantizer with $N = 3$. The solid line in the left plot is the trajectory of the system (starting at $\boldsymbol{x}\,(0) = [1; 0]$). The dotted line in that plot is the state estimate. The dashed-dot lines represent the jumps in the state estimate after a new measurement is received. The top right plot shows the first 15 seconds of the measured output ($T_s = 1s$). The vertical dotted lines depict the only one bounded quantization region. The controller is in the *capture* mode where these vertical lines are bounded by arrows facing outward; in the *update* mode where the arrows are facing inward; and in the *detect* mode where the vertical lines are bounded by small horizontal lines. The bottom right plot shows 15 seconds of the steady-state behavior of the simulation, where an escape of the trajectory due to disturbances is detected at $t = 119s$, and then the trajectory is recaptured at $t = 122s$. The pseudo-inverse observer (see §5.1) was used in this simulation. The other design parameters were: $d = 6$, $\mu\,(0) = 0.25$, $\Omega_{out} = 2$, $\alpha = 0.02$, $s = 0.05$, $K = [0.6, -1.5]$. The disturbances followed the zero-mean normal distribution with standard deviation of 0.2.

To see that indeed this condition is sufficient note first that from (6):

$$F(\mu;k;k-r) = \max_{i\in\{1\dots n_x\}} \sum_{l=k-2r}^{k-r-1} \sum_{m=1}^{n_y} \left|(CA_dG_0)_{i,(l+2r-k)n_y+m}\right| \mu(r+l)$$
$$\leq \left(\max_{i\in\{1\dots n_x\}} \sum_{m=1}^{n_y r} \left|(CA_dG_0)_{i,m}\right|\right) \|\mu\|_{k-r\dots k-1} = \|CA_dG_0\| \, \|\mu\|_{k-r\dots k-1},$$

so that

$$\frac{F(\mu';k;k-r)+\alpha}{N} \leq \sigma_{pi} \|\mu'\|_{k-r\dots k-1} + \frac{\alpha}{N}. \tag{12}$$

Assume d is a multiple of r and α satisfies $\sigma_{pi} + \frac{a}{N} \leq 1$ so that $\forall l \in \mathbb{N}$: $\sigma_{pi}^{l+1} + \sum_{m=0}^{l} \sigma_{pi}^{m} \frac{\alpha}{N} \leq \sigma_{pi}^{l} + \sum_{m=0}^{l-1} \sigma_{pi}^{m} \frac{\alpha}{N}$. With these assumptions, and from (12) we have by induction that for all $l \in \{1\dots d/r-1\}$:

$$\|\mu'\|_{lr\dots(l+1)r-1} \leq \sigma_{pi}^{l} + \sum_{m=0}^{l-1} \sigma_{pi}^{m} \frac{\alpha}{N} \doteq V(l) \qquad \text{and}$$
$$\|\mu'\|_{d-r\dots d-1} \leq \max\left\{ \frac{N}{N-2}\sigma_{pi} V(d/r-1) + \frac{\alpha}{N-2}, \right.$$
$$\left. \left(\frac{N}{N-2}\sigma_{pi}\right)^{r} V(d/r-1) + \sum_{m=0}^{r-1} \left(\frac{N}{N-2}\sigma_{pi}\right)^{m} \frac{\alpha}{N-2} \right\}$$

It can now be easily seen that by taking d to be large enough, and α to be small enough, we can make $\|\mu'\|_{d-r\dots d-1} < 1$ which satisfies the convergence property.

5.2 Luenberger-Type Observer

Another commonly used observer for unquantized, output feedback systems is the Luenberger observer:

$$\hat{\boldsymbol{x}}(k+1) = A_d\hat{\boldsymbol{x}}(k) + \boldsymbol{u}_d(k) + L\left(\boldsymbol{y}(k) - C\hat{\boldsymbol{x}}(k)\right), \tag{13}$$

where $L \in \mathbb{R}^{n_x \times n_y}$ is chosen so that $A_d - LC$ is Schur[5]. Given that A_d and C are an observable pair, such an L is guaranteed to exist. Since the Luenberger observer requires some initialization, we can use G_0 as in the pseudo-inverse observer (10). We can then replace in the algorithm all the computations of the state estimate, $\hat{\boldsymbol{x}}_u(k) = G(\boldsymbol{z};\boldsymbol{u}_d;k;p)$, when $p > 0$, with

$$\hat{\boldsymbol{x}}_u(k) = \hat{\boldsymbol{x}}_p(k) + L\left(\boldsymbol{z}(k-1) - C\hat{\boldsymbol{x}}_u(k-1)\right).$$

Using this alternative is equivalent to using

$$G_i = \left[\, * \mid (A_d - LC)^{i-2} L \mid \dots \mid L \mid 0_{n_x\times n_y} \right], \qquad i \in \{1\dots d-1\} \tag{14}$$

[5] All the eigenvalues of a Schur matrix are inside the unit ball on the complex plane. This is the discrete counterpart to a Hurwitz matrix

where

$$* \doteq (A_d - LC)^i \, G_0 + \left[0_{n_x \times (r-1) n_y} \mid (A_d - LC)^{i-1} \, L \right] .$$

Remark 1. This observer will satisfy the first requirement from §4.1. However, we have not been able yet to derive an easily verifiable sufficient condition for the second requirement as we did for the pseudo-inverse observer with (11). Therefore, to verify that such an observer satisfies the second requirement, one has to generate the μ''s according to (5) using (14), and then verify that (7) holds.

Remark 2. The standard formulation for a Luenberger observer is (13). However, note that when we need to construct $\hat{\boldsymbol{x}}_u(k+1)$, on which the control inputs from $t = (k+1)\,T_s$ to $t = (k+2)\,T_s$ are based, we already have the measurement $\boldsymbol{z}\,(k+1)$. Therefore, instead of (13) it will be better to use

$$\hat{\boldsymbol{x}}(k+1) = A_d \hat{\boldsymbol{x}}(k) + \boldsymbol{u}_d(k) + L\left(\boldsymbol{y}(k+1) - C\left(A_d \hat{\boldsymbol{x}}(k) + \boldsymbol{u}_d(k)\right)\right),$$

which requires that $A_d - LCA_d$ is Schur. With these settings (14) becomes:

$$G_i = \left[(A_d - A_d LC)^i \, G_0 \mid (A_d - A_d LC)^{i-1} \, L \mid \ldots \mid L \right], \qquad i \in \{1 \ldots d-1\}.$$

6 Conclusion

In this paper we showed how to implement a stabilizing controller when only a partial subspace of the state space is measured, and furthermore the measurements are quantized with a finite number of quantization regions. The controller is also robust, in the ISS sense, to unknown disturbance which can be injected to the system. In our design, we allow flexibility in designing the observer and the stabilizing control law, thus allowing further balancing between ease of implementation and performance.

As mentioned in the introduction, this paper extends the results in [15] from the state-feedback scenario to the output-feedback scenario. Future developments will be to extend the results further to systems with delays and nonlinear systems.

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