Stabilization of Linear Systems under Coarse Quantization and Time Delays

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Abstract: We consider the problem of stabilizing a control system using a coarse state quantizer in the presence of time delays. We assume the quantizer has an adjustable “center” and “zoom” parameters, and employ an alternating “zoom out” /“zoom in” mechanism in order to achieve a large region of attraction while having the system converge to a small region around the origin. This mechanism is adopted from our previous work where delays were not considered. Here we show that the control system, using the same mechanism and without making any changes in order to accommodate delays explicitly, remains stable under small delays. The main tool we use to prove the result is the nonlinear small-gain theorem.

Keywords: Quantization, Delays, Stability

1. INTRODUCTION

Networked control systems are characterized by simultaneous presence of several communication constraints, such as quantization, time scheduling, time delays, packet dropouts, interference, and so on. While early work focused on studying just one of these aspects, more recently results able to handle two or more are beginning to emerge (see, e.g., Heemels et al. (2009) and the references therein). In this paper we address two of the phenomena mentioned above, namely, state quantization and time delays.

The approach described here has its roots in two related lines of recent work. The first relevant contribution is the method for stabilizing nonlinear systems with quantization and delays presented in Liberzon (2006). The analysis in Liberzon (2006) centers around the concept of input-to-state stability (ISS) and an associated small-gain theorem, and is based on the approach of Teel (1998). An important drawback of the result given in Liberzon (2006), however, is that it does not attempt to minimize the data rate and so the bound on the number of quantization regions that it requires is very conservative. On the other hand, there have been many results on quantized stabilization with minimal data rate. In the context of nonlinear systems, an ISS control framework was developed in Liberzon and Hespanha (2005) and subsequently refined in Sharon and Liberzon (2010) to obtain ISS with respect to external disturbances. However, these results do not allow the presence of time delays.

Thus, the contribution of the present work is essentially to extend the method of Sharon and Liberzon (2010) to the case where (possibly time-varying) delays are present in addition to state quantization. Although we assume there are no external disturbances in this paper, we do rely on the ISS property which we established in Sharon and Liberzon (2010) after we show that error signals which arise due to delays can be regarded as external disturbances. The ISS small-gain analysis employed in this paper is similar in spirit to that used in Liberzon (2006), but it becomes more challenging due to the dynamics of the quantizer which are necessary to achieve minimal data rate (in Liberzon (2006) only a static quantizer was considered). We believe that, by virtue of being able to handle both quantization and delays while enforcing a minimal data rate, our result will be of greater use for analysis and design of networked control systems. In this paper we consider linear plant dynamics, but our method is nonlinear in nature and we expect it to naturally extend to suitable nonlinear systems along the lines of (Sharon and Liberzon, 2010, Section VI).

Among other noteworthy references dealing with quantization and delays, using approaches different from ours, we mention Fridman and Dambrine (2009), De Persis and Mazenc (2009), and Sailer and Wirth (2009). The first two of these papers employ Lyapunov-Krasovskii functionals for linear and nonlinear systems, respectively, while the last one handles nonlinear systems by sending time information along with the encoded state.

The outline of this paper is as follows: In §2 we define the quantized and delayed control system which we address in this paper; in §3 we recall the controller we developed in Sharon and Liberzon (2010); in §4 we present the main result of this paper; §5 is dedicated to proving the main result; concluding remarks are in §6.

2. PROBLEM FORMULATION

The system we consider consists of three components: the plant, the quantizer, and the controller. The continuous-time plant we are to stabilize is as follows (t ∈ R≥0):

\[ \dot{x}(t) = Ax(t) + Bu(t) \] (1)

where \( x \in \mathbb{R}^n \) is the state and \( u \in \mathbb{R}^m \) is the control input.
The quantizer samples the state of the plant every \( T_s \) seconds and generates the information for the controller, \( z: \{ k T_s | k \in \mathbb{Z}_{\geq 0} \} \to \mathbb{R}^n: \)

\[
z(k T_s) = Q(\mathbf{x}(k T_s); \mathbf{c}(k T_s), \mu(k T_s)),
\]

where \( \mathbf{c}: \{ k T_s | k \in \mathbb{Z}_{\geq 0} \} \to \mathbb{R}^n \) and \( \mu: \{ k T_s | k \in \mathbb{Z}_{\geq 0} \} \to \mathbb{R}_{>0} \) are quantization parameters and \( Q \) is the quantization function. For convenience we will use the notation \( z_k \equiv z(k T_s) \), and similarly for other variables.

We will present our results using the following (square) quantizer. We assume \( N \) is an odd number, \( N \geq 3 \), which counts the number of quantization regions per state dimension. The quantizer is denoted by \((Q_1, \ldots, Q_n)^T = Q(\mathbf{x}; \mathbf{c}, \mu)\) where each scalar component is defined as follows:

\[
Q_i(\mathbf{x}; \mathbf{c}, \mu) \equiv c_i + 2 \mu \times \begin{cases} (-N + 1)/2 & x_i - c_i \leq (-N + 2) \mu \\
(N - 1)/2 & (N - 2) \mu < x_i - c_i \\
\lfloor (x_i - (c_i + \mu))/(2\mu) \rfloor & \text{otherwise}
\end{cases}
\]  

(3)

We will refer to \( \mathbf{c} \) as the center of the quantizer, and to \( \mu \) as the zoom factor. Note that what will actually be transferred from the quantizer to the controller will be an index to one of the quantization regions. The controller, which either generates the values \( \mathbf{c} \) and \( \mu \) and shares them with the quantizer or knows the rule by which they are generated, will use this information to convert the received index to the value of \( Q \) as given in (3). This setup is the same as in Sharon and Liberzon (2010).

Due to delays, for every \( k \in \mathbb{Z}_{\geq 0} \) the controller receives the information \( z_k = z(k T_s) \) only at time \( k T_s + \delta_k \) where \( \delta_k \in [0, T_s) \) is the delay. The delay is unknown to the controller and it does not need to be fixed. We set \( \delta_{\text{max}} = \sup_{k \geq 0} \delta_k \).

In this paper we will use the \( \infty \)-norm unless otherwise specified. For vectors, \( \|\mathbf{x}\| \equiv \|\mathbf{x}\|_{\infty} \equiv \max_i |x_i| \). For continuous-time signals, \( \|\mathbf{w}\|_{(t_1, t_2)} \equiv \max_{\mathbf{t} \in [t_1, t_2]} \|\mathbf{w}(\mathbf{t})\|_{\infty}, \) \( \|\mathbf{w}\|_{[0, \infty)} \equiv \sup_{t \geq 0} \|\mathbf{w}(t)\|_{\infty}. \) For discrete-time signals, \( \|\mathbf{z}\|_{(k_1, k_2)} \equiv \max_{k \in [k_1, k_2]} \|z_k\|, \) \( \|\mathbf{z}\|_{(0, \infty)} \equiv \sup_{k \geq 0} \|z_k\|. \) For matrices we use the induced norm corresponding to the specified norm (\( \infty \)-norm if none specified). For piecewise continuous signals we will use the superscripts \( + \) and \( - \) to denote the right and left continuous limits, respectively: \( x_k^+ \equiv x^+(k T_s) \equiv \lim_{t \downarrow k} x(k T_s + t), \) \( x_k^- \equiv x^-(k T_s) \equiv \lim_{t \uparrow k} x(k T_s + t). \)

3. CONTROLLER DESIGN

We implement the same controller as in Sharon and Liberzon (2010). One of the tasks of the controller is to generate the state estimate, \( \hat{\mathbf{x}}(t) \), for which we will use the notation \( \hat{\mathbf{x}}_k \equiv \hat{\mathbf{x}}(k T_s + \delta_k) \). The controller keeps and updates a discrete time step variable, \( k \in \mathbb{N} \), whose value will correspond to the current sampling time of the continuous system. When a new measurement is produced at times \( k T_s \), it may be used to update the state estimate \( \hat{\mathbf{x}}_k \) where \( k \) is the discrete time step. At each discrete time step, the controller will operate in one of three modes: capture, measurement update or escape detection. The current mode will be stored in the variable \( \text{mode}(k) \in \{\text{capture, update, detect}\} \). The controller will also use \( p_k \in \mathbb{Z} \) and \( \text{saturated}(k) \in \{\text{true, false}\} \) as auxiliary variables. The variable \( p_k \) counts the number of sampling times at which the controller was in the measurement update mode since the last sampling time at which it was either in the capture mode or the escape detection mode. We note that the difference between the measurement update mode and the escape detection mode is that in the former we set the quantizer so as to minimize the estimation error, but this comes at the expense of not being able to detect saturation.

We assume the control system is activated at \( k = 0 \) (\( t = 0 \)). We initialize \( \hat{\mathbf{x}}(0) = 0, \text{mode}(0) = \text{capture}, \) \( p_0 = 0, \) and \( \mu_0 = s, \) where \( s \) is a positive constant which will be regarded as a design parameter. We also use the following design parameters: \( \alpha \in \mathbb{R}_{>0}, \Omega_{\text{out}} \in \mathbb{R} \) such that \( \Omega_{\text{out}} > \|e_0^T A\|, \) and \( P \in \mathbb{Z} \) such that \( P \geq 1 \). We refer the reader to (Sharon and Liberzon, 2007, §V) for a detailed qualitative discussion on how each design parameter affects the system performance. The last design parameter is the static feedback control law, \( K \), which should be chosen such that \( A + BK \) is Hurwitz.

On the time interval between the arrivals of new measurements, \( t \in [k T_s + \delta_k, (k + 1) T_s + \delta_{k+1}] \), the controller continuously updates the state estimate and the control input based on the nominal system dynamics:

\[
\dot{\hat{\mathbf{x}}}(t) = A \hat{\mathbf{x}}(t) + B u(t) \quad u(t) = K \hat{\mathbf{x}}(t).
\]

Whenever a new measurement is received from the quantizer at time \( k T_s + \delta_k \), the controller executes sequentially Algorithm 1–Algorithm 5:

\[\text{Algorithm 1 Preliminaries}\]

\[
\text{if } \exists i \text{ such that } (z_k)_i = (c_k)_i \pm (N - 1) \mu_k \text{ then set } \text{saturated}(k) = \text{true} \]

\[
\text{else set } \text{saturated}(k) = \text{false} \]

\[
\text{if mode}(k + 1) = \text{mode}(k) \]

\[\text{Algorithm 2 capture mode}\]

\[
\text{if mode}(k) = \text{capture then set } p_k = 0 \]

\[
\text{if not } \text{saturated}(k) \text{ then update the state estimate: } \hat{\mathbf{x}}(k T_s + \delta_k) = z_k \text{ set mode}(k + 1) = \text{update} \]

\[
\text{end if} \]

\[\text{Algorithm 3 measurement update mode}\]

\[
\text{if mode}(k) = \text{update then set } p_k = p_{k - 1} + 1 \]

\[
\text{update the state estimate: } \hat{\mathbf{x}}(k T_s + \delta_k) = z_k \text{ if } p_k = P - 1 \text{ then set mode}(k + 1) = \text{detect} \]

\[
\text{end if} \]

4. MAIN RESULT

Let \( \mu'_0 = 1, \mu_k = (\|e_0^T A\| \mu'_{k - 1} + \alpha) / N, k = 1, \ldots, P - 1, \mu'_P = (\|e_0^T A\| \mu'_{P - 1} + \alpha) / (N - 2) \). If \( \mu'_P < 1 \) then we say that the design parameter \( \alpha \) satisfies the convergence property. In (Sharon and Liberzon, 2010, Lemma 1) we
Algorithm 4 escape detection mode

if mode(k) = detect then
    if not saturated(k) then
        update the state estimate: \( \hat{x}(kT_s + \delta_k) = z_k \)
        set \( p_k = 0 \)
        set mode(k + 1) = update
    end if
else
    set \( p_k = 0 \) and \( \mu_k = s \)
    switch to capture mode: set mode(k + 1) = capture
end if

Algorithm 5 preparing for next sampling

if mode(k + 1) = capture then
    set \( \mu_{k+1} = \Omega_{out} p_k \)
else if mode(k + 1) = update then
    set \( \mu_{k+1} = (|[A]_p|\|p_k + \alpha\|p_k-p_p|)(N) \)
else if mode(k) = detect then
    set \( \mu_{k+1} = (|[A]_p|\|p_k + \alpha\|p_k-p_p|)(N - 2) \)
end if
set \( e_k+1 = \exp(T_s(A + BK))\hat{x}(kT_s + \delta_k) \)

proved that a necessary and sufficient condition for the existence of such an \( \alpha \) is \( \|e^{T_sA}\|/N < 1 \).

Theorem 1. Given an implementation of the controller above with any valid choice for the design parameters such that \( \alpha \) satisfies the convergence property, the closed loop system will have the following semiglobal stability property: For every \( x_{max} \geq 0 \), there exists a small but strictly positive \( \delta_{max} \) such that if \( \delta_{max} \leq \delta_{max} \) then the following bound, \( \forall t \geq 0 \):

\[
|x(t)| \leq \beta_\alpha \left( |x(0)|, t \right) + \gamma \left( \delta_{max} \right) \tag{5}
\]

holds whenever \( |x(0)| \leq x_{max} \), where the function \( \beta_\alpha \) is of class \( KL_1 \) (\( \beta \in KL \)) and \( \gamma \) is of class \( K \) (\( \gamma \in K \)).

Remark: Known results on delays, Liberzon (2006) for example, provide what can be interpreted as a more general result than (5), in which the time 0 is replaced with \( t_0 \) and the bound holds for arbitrary \( t_0 \). In fact, an intermediate step in proving Theorem 1 (see (31) below) does provide a similar result which holds for arbitrary \( t_0 \).

We start with a technical lemma:

Lemma 2. Let a system with state \( x \) satisfy the following relation, \( \forall t \geq t_0 \geq 0 \):

\[
|x(t)| \leq \beta_x \left( |x(t_0)|, t - t_0 \right) + \gamma_x \left( \|x_d(t_0, t)\|_{\gamma_\omega'}, \gamma_w \|w(t_0, t)\|_{\gamma_\omega'} \right) \tag{6}
\]

where \( \beta_x \in KL_1 \), and \( \gamma_x, \gamma_w \in K_{\infty} \). If \( \gamma_z(< \gamma') \) for some \( \lambda \), then for every function \( \gamma \in K_{\infty} \) such that

\[
\gamma(\nu) \geq \left( 1 + \sqrt{\lambda} \right) \left( 1 + \lambda \left( 1 + \sqrt{\lambda} \right) \right) \gamma_w(\nu) \tag{7}
\]

there exists a function \( \beta \in KL_1 \) such that \( \forall t \geq t_0 \geq \Delta \):

\[
|x_{d}(t)| \leq \beta \left( \|x_0(t_0)\|, t - t_0 \right) + \gamma \left( \|w\|_{\gamma_w(t_0, t)} \right). \tag{8}
\]

Proof. First we have \( \forall t \geq t_0 \geq \Delta \):

\[
|x_d(t)| \leq \beta_d \left( \|x_0(t_0)\|, t - t_0 \right) + \gamma_d \left( \|w\|_{\gamma_w(t_0, t)} \right)
\]

where \( \beta_d(\nu, t) = 1_{t < \Delta} \nu + 1_{t \geq \Delta} e^{-1/(t-\Delta)} \) with arbitrary \( \epsilon > 0 \) (1_{t < \Delta} is the characteristic function whose value is 1 if \( t < \Delta \) and 0 otherwise) and \( \gamma_d(\nu) = \nu \). Note that \( \beta_d \in KL \) and \( \gamma_d \in K_{\infty} \). Defining \( y(t) \doteq \gamma_z(\|x_d(t)\|) \) we can have \( \forall t \geq t_0 \geq \Delta \):

\[
|y(t)| \leq \beta_y \left( |y(t_0)|, t - t_0 \right) + \gamma_y \left( \|w\|_{\gamma_w(t_0, t)} \right)
\]

where \( \beta_y(\nu, t) = \gamma_z \left( \beta_y(z_{t-1}^{-1}(\nu), t) \right) \in KL \) and \( \gamma_y(\nu) = \gamma_x(\nu) \in K_{\infty} \).

Invoking the Small-Gain Theorem (Jiang et al., 1994, Theorem 2.1), with \( \beta_1(\nu, t) = \beta_y(\nu, t), \gamma_1(\nu) = \nu \), \( z_{t-1}^{-1}(\nu) = \gamma_w(\nu), \beta_2(\nu, t) = \beta_d(\nu, t), \gamma_2(\nu) = \gamma_x(\nu), \gamma_2(\nu) = 0 \), and \( p_1 = p_2 = 1/\sqrt{\lambda} - 1 \), we can get functions \( \beta_0, \gamma_0 \in KL \) such that \( \forall t \geq t_0 \geq \Delta \):

5. PROOF

We start with a brief overview of the proof. In addition to the state signal, \( x(t) \), we define a state estimation error signal, \( \hat{x}(t) - x(t) \) (the explicit dependence of \( \delta \) on \( t \) will be provided in the proof itself). We also define two additional signals, \( \theta_x(t) = x(t) - \hat{x}(t) \) and \( \theta_x(t) = \hat{x}(t) - \hat{x}(t) \). We use a small-gain argument between \( x \) and \( \theta_x \) in Lemma 3 to show that for a sufficiently small delay, there exists an ISS relation between the
\[ |x(t)| \leq \beta' \left( \gamma_x(|x_0|), |t - t_0| \right) + \gamma \left( \|w\|_{[t_0,t]} \right) \leq \beta'' \left( |x_0|, |t - t_0| \right) + \gamma \left( \|w\|_{[t_0,t]} \right) \]

for every \( \gamma \in \mathcal{K}_{\infty} \) which satisfies (7). Because it must hold that \( \beta''(\nu,0) \geq \nu \) we can arrive at (8) with \( \beta(\nu,t) = \beta''(\nu,\max(0,t - \Delta)) \). \( \square \)

Define \( k(t) \) as max \( \{k \in \mathbb{Z}_{\geq 0} | KT_s + \delta_k \leq t \} \), the index of the last sampling which arrived at the controller before time \( t \). With this definition we can write:

\[ \dot{x}(t) = Ax(t) + BK(x(t - \delta_k(t)) + \ddot{x}(t)) = (A + BK)x(t) + BK(\theta_x(t) + \ddot{x}(t)) \]  

(9)

where \( \theta_x(t) = x(t - \delta_k(t)) - x(t) \) and \( \ddot{x}(t) = \dot{x}(t) - x(t - \delta_k(t)) \).

Lemma 3. There exists a sufficiently small, but strictly positive, \( \delta_{\max} \), such that if \( \delta_{\max} \leq \delta_{\max} \) then the following ISS relation, \( \forall t \geq t_0 \geq \Delta \):

\[ |x_k(t)| \leq \tilde{\beta}_x(|x_0|), |t - t_0| + \gamma \left( |x|_{[t_0,t]} \right) \]

holds where \( \beta_x \in \mathcal{KL} \) and \( \gamma_x \in \mathcal{K}_{\infty} \) is a linear function.

Proof. A standard result on ISS for linear systems is that the system defined by (9) follows

\[ |x(t)| \leq \tilde{\beta}_x(|x_0|), |t - t_0| + \gamma \left( |x|_{[t_0,t]} \right) \]

where \( \tilde{\beta}_x \in \mathcal{KL} \) and \( \gamma_x \in \mathcal{K}_{\infty} \) is a linear function. For example one can take \( \beta_x(\nu,t) = c e^{-\sigma t} \nu \) and \( \gamma_x(\nu) = \frac{\|A+BK\|}{\sigma} \nu \) where \( c > 0 \) and \( \sigma > 0 \) are such that \( \|e^{(A+BK)t}\| \leq c e^{\sigma t} \) \( \forall t \geq 0 \).

We also have from the first line in (9), \( \forall t \geq \Delta \):

\[ |\theta_x(t)| = \tilde{\beta_x} \left( |x_0|, |t - t_0| + \gamma \left( |x|_{[t_0,t]} \right) \right) \leq \beta_{\max} \left( ||A| + ||B|| \right) |\tilde{x}(t - \delta_k(t) + \gamma \left( |x|_{[t_0,t]} \right) \right) + \beta_{\max} \left( ||A| + ||B|| \right) ||x_0||_{[t_0,t]} + \gamma \left( |x|_{[t_0,t]} \right) \]

(12)

For the last inequality we use the fact that \( \Delta \geq 2 \delta_{\max} \). Substituting this into (11), we get (6) with

\[ \gamma_x(\nu) = \tilde{\gamma}_x \left( \tilde{\beta_x} \right), \gamma_w(\nu) = \tilde{\gamma}_w \left( \tilde{\beta_x} \right) \]

(we used the fact that \( \gamma_x \) is a linear function). Choosing \( \delta_{\max} \) such that \( \tilde{\gamma}_x \left( \tilde{\beta_x} \right) \leq \nu \) \( \forall \nu \), (10) follows by Lemma 2. \( \square \)

Define \( \ddot{k}(t) = \lfloor t/T_s \rfloor \). Another way to expand (9) is as follows, \( \forall t \geq \Delta \):

\[ \dot{x}(t) = Ax(t) + Bu(t) = Ax(t) + BK \dot{x}(t) = \begin{cases} Ax(t) + BK \dot{x}(t) \mid \theta_x(t) = \dot{x}(t) + \delta_k(t) \end{cases} \]

(13)

where \( \theta_x(t) = \dot{x}(t - \delta_k(t)) - \dot{x}(t) \). For \( t \geq T_s + \delta_1 \), \( t \neq kT_s + \delta_k \) \( \forall k \), the state estimate evolves according to (4) and thus the estimation error, \( \ddot{x} \), evolves according to

\[ \ddot{x}(t) = \dot{x}(t) - \dot{x}(t - \delta_k(t)) = A\ddot{x}(t) - BK(\theta_{\dot{x}}(t) + \theta_x(t - \delta_k(t))) \]

(14)

Denoting \( \ddot{w}(t) = BK(\theta_v(t) + \theta_x(t - \delta_k(t))) \),

\[ w_k^d = \begin{cases} \ddot{w}(t) \mid t \leq (k+1)T_s + \delta_k \end{cases} \]

we have that \( \forall k \geq k_0 \):

\[ \ddot{x}_k(t) = \ddot{x}(t) - c((k+1)T_s - x((k+1)T_s)) + e^{T_sA} \ddot{x}(kT_s + \delta_k) + e^{T_s\delta_k} \ddot{x}_k^d \]

(15)

where \( c \) is the quantization parameter defining the center of the quantizer. In (Sharon and Liberzon, 2010, Proposition 2) we proved that if the system satisfies (15), then the following holds:

Corollary 4. There exist functions \( \beta_{\dot{x},d} \in \mathcal{KL} \) and \( \gamma_{\dot{x},d} \in \mathcal{K}_{\infty} \) such that \( \forall k \geq k_0 \geq 1 \):

\[ |\ddot{x}_k(t)| \leq \beta_{\dot{x},d} \left( |\ddot{x}_k(t_0)|, |t - t_0| + \gamma_{\dot{x},d} \left( |\ddot{x}_k(t_0)| \right) \right) \]

(16)

The function \( \psi(\cdot,\cdot) \) as a function of its first argument when its second argument is fixed, is continuous, non-decreasing and non-negative. As a function of its second argument when its first argument is fixed, it is continuous.

Lemma 5. The delayed estimation error, \( \ddot{x}_k(t) \), satisfies the following condition, \( \forall t \geq T_s + \Delta \):

\[ |\ddot{x}_k(t)| \leq \beta_{\dot{x}} \left( |\ddot{x}_k(t_0)|, |t - t_0| + \delta_k \right) + \gamma_{\dot{x}} \left( |\ddot{x}_k(t_0)| \right) \]

(17)

where \( \beta_{\dot{x}} \in \mathcal{KL} \) and \( \gamma_{\dot{x}}, \gamma_{\dot{w}} \in \mathcal{K}_{\infty} \).

Proof. We can bound \( w_k^d \), \( \forall k \geq 1 \), as

\[ w_k^d \leq e^{T_s\|A\|} \|BK\| \int_{kT_s + \delta_k}^{(k+1)T_s + \delta_k} |\theta_{\dot{x}}(t)| dt + e^{T_s\|A\|} \|BK\| \int_{kT_s + \delta_k}^{(k+1)T_s + \delta_k} \|\theta_x(t)\| dt \]

We can also bound the estimation error between updates, \( \forall k \geq 1 \) and \( \forall t \in [kT_s + \delta_k, (k+1)T_s + \delta_k] + 1\):

\[ |\ddot{x}(t)| \leq e^{(T_s + \delta_{\max})\|A\|} |\ddot{x}| + e^{(T_s + \delta_{\max})\|A\|} \int_{kT_s + \delta_k}^{(k+1)T_s + \delta_k} |\theta_{\dot{x}}(t)\| dt + e^{(T_s + \delta_{\max})\|A\|} \|BK\| \int_{kT_s + \delta_k}^{(k+1)T_s + \delta_k} \|\theta_x(t)\| dt \]

Combining these two bounds with (16) and the first inequality in (12), we can arrive at, \( \forall t \geq \Delta \):

\[ |\ddot{x}(t)| \leq \beta_{\dot{x}} \left( |\ddot{x}_k(t_0)|, |t - t_0| + \delta_k(t_0) \right) + \gamma_{\dot{x}} \left( \beta_{\dot{x}} \left( |\ddot{x}_k(t_0)|, |t - t_0| + \delta_k(t_0) \right) \right) \]

(18)

where \( \beta_{\dot{x}} \in \mathcal{KL} \) and \( \gamma_{\dot{x}}, \gamma_{\dot{w}}, \gamma_{\dot{w}} \in \mathcal{K}_{\infty} \).
From the definition of $\theta_c$, $\forall t \geq \min \{2\delta_0, T_s + \delta_1\}$:

$$\theta_c(t) = -\int_{t-d(t)}^{t} \bar{x}(t) \, dt - \sum_{t \in (t-d(t), t]} (\bar{x}(t) - \bar{x}^{-}(t))$$

(19)

where $\chi = \{ t \geq 0 \mid \exists k \in \mathbb{N} \text{ such that } \tau = kT_s + \delta_k \}$. Each $t \in \chi$ affects $\theta_c$, through the second term on the right-hand side of (19) only in a time interval of length at most $\delta_{\max}$. The set $(kT_s + \delta_k - \delta(kT_s + \delta_k), (k+1)T_s + \max \{ \delta_k, \delta_{k+1} \}) \cap \chi$ contains at most two elements $\forall k \geq 1$. Using also (14), we can finally arrive at the bound: $\forall k \geq 2$ and $\forall t \in [kT_s + \delta_k, \max \{ (k+1)T_s + \delta_k, (k+1)T_s + \delta_{k+1} \}):

$$\int_{kT_s + \delta_k}^{t} |\theta_c(t)| \, dt \leq 4\delta_{\max} \|\bar{x}\|_{[kT_s, t]} + \delta_{\max} (T_s + \delta_{\max}) (\|A - BK\| + \|BK\|) \|\bar{x}\|_{[kT_s - \delta_{k-1}, t]} + \delta_{\max} (T_s + \delta_{\max}) 2 \|BK\| \|x\|_{[kT_s - \delta_{k-1} - \delta(kT_s - \delta_{k-1}), t - \delta_{k+1}]}.$$

(20)

Using (20) in (18) and the same argument we used at the end of the proof of Lemma 2 to move from a bound on $|x(t)|$ to a bound on $|\bar{x}(t)|$, we can arrive at the result stated in the lemma. \hfill \square

A corollary of (Sharon and Liberzon, 2010, Theorem 4) gives us the following:

**Corollary 6.** Assume that (17) holds, and that there exist $r_1 > r_0 \geq 0$, and $\lambda < 1$ such that $\forall t \in [r_0, r_1]$:

$$\gamma_e (\delta_{\max}; \mu(k \delta_1)) \leq \lambda r$$

and

$$\frac{1}{1 - \lambda} (\beta_e (|\bar{x}(t)|, 0; \mu(k \delta_1)) + \gamma_{\bar{w}} (\delta_{\max}; \mu(k \delta_1)) < r_1. $$

(22)

Then $|\bar{x}(t)|_{[\delta_{\min}, \infty]} < r_1$.

A corollary of the Small-Gain Theorem (Jiang et al., 1994, Theorem 2.1) gives us the following local result:

**Corollary 7.** Let $x_1, x_2, w$ be three signals satisfying $\forall t \geq 0$:

$$|x_1(t)| \leq \beta_1 (|x_1(t_0)|, t - t_0) + \gamma_{1,x} (\|x_2\|_{[t_0, t]} + d_1)$$

$$|x_2(t)| \leq \beta_1 (|x_2(t_0)|, t - t_0) + \gamma_{2,x} (\|x_1\|_{[t_0, t]} + d_2)$$

where $\beta_1, \beta_2 \in \mathcal{K} \mathcal{L}, \gamma_{1,x}, \gamma_{1,w}, \gamma_{2,x}, \gamma_{2,w} \in \mathcal{K}$ and $d_1, d_2 \geq 0$. Assume that for some $r_1 > r_0 \geq 0$ the small-gain condition

$$\gamma_{1,x} (\gamma_{2,x} (r)) < r, \quad \forall r \in [r_0, r_1]$$

holds and it can be guaranteed that $\|x_1\|_{[0, \infty]} < r_1, \|x_2\|_{[0, \infty]} < \gamma_{1,x}^{-1} (r_1)$. Then we can get that $\forall t \geq 0$:

$$|x_1(t)| \leq \beta (x_1(t_0), t - t_0) + \gamma (\gamma_{1,w} (\|w\|_{[t_0, t]})) + d$$

$$|x_2(t)| \leq \beta (x_2(t_0), t - t_0) + \gamma (\gamma_{2,w} (\|w\|_{[t_0, t]})) + d$$

where $\beta \in \mathcal{K} \mathcal{L}$ and $\gamma \in \mathcal{K}$. Furthermore in the limit as $d_1 \to 0, d_2 \to 0, r_0 \to 0$, we get $d = 0$.

With these two corollaries we derive the following lemma:

**Lemma 8.** For any $d' > 0, x'_{\max}$, $\bar{x}_{\max}$ and $\mu_{\max}$ there exists a sufficiently small, but strictly positive, $\delta_{\max}$, such that if $\delta_{\max} \leq \delta_{\max}$ then the following ISS relation, $\forall t \geq t_0 \geq \Delta$:

$$|\bar{x}(t)| \leq \beta_e (|\bar{x}(t_0)|, t - t_0) + \gamma_e \left( \delta_{\max}, \|\bar{x}\|_{[t_0, t]} \right) + d'$$

(23)

where $\beta_e \in \mathcal{K} \mathcal{L}$ and $\gamma_e$ holds for all $\forall |\bar{x}(\Delta)| \leq x'_{\max}$, $\forall |\bar{x}(\Delta)| \leq \bar{x}_{\max}$, and $\forall \mu(k \Delta) \leq \mu_{\max}$. Furthermore, for $\forall \delta_{\max} \leq \delta_{\max}$, we can write

$$\|\bar{x}(\Delta)\| \leq \gamma_1 (\delta_{\max}) + \gamma_e (\bar{x}(\max) \delta_{\max})$$

(24)

where

$$\lim_{\delta_{\max} \to 0} \gamma_1 (\delta_{\max}) = \sup_{\mu \in [0, \mu_{\max}]} \beta_e (x'_{\max}, 0; \mu)$$

(25)

**Proof.** We first note that for any $x'_{\max} \geq 0$, $\bar{x}_{\max} \geq 0$, $\mu_{\max} \geq 0, r_1 > \max_{\mu \in [0, \mu_{\max}]} \beta_e (x'_{\max}, 0; \mu)$, and $r_0 \in (0, r_1)$, one can find $\delta_{\max} > 0$ and $\lambda < 1$ for which the assumptions in Corollary 6 are satisfied $\forall |\bar{x}(\Delta)| \leq x'_{\max}$, $\forall |\bar{x}(\Delta)| \leq \bar{x}_{\max}$, and $\forall \mu(k \Delta) \leq \mu_{\max}$. Taking $\delta_{\max}$ to be smaller if necessary, we can also have

$$\gamma'(\delta_{\max} < r) \quad \forall r \in [r_0, r_1].$$

We can now use the local version of the Small-Gain Theorem (Corollary 7), similarly to how we used the Small-Gain Theorem in Lemma 2, and arrive at (23).

Note that when applying Corollary 7 to (26), we will have $d_1 = 0$ and $d_2 = 0$. Thus we get that $\lim_{r_0 \to 0} d' = 0$. And since we can choose $r_0$ to be arbitrarily small by reducing $\delta_{\max}$, we can in turn make $d'$ arbitrarily small. Assume now that (21) holds for some $\delta_{\max} = \delta_{\max}$. Then we can replace the constant $\lambda$ in (21) with a function $\lambda (\delta_{\max})$ such that (21) still holds for every $\delta_{\max} \leq \delta_{\max}$, and furthermore, $\lim_{\delta_{\max} \to 0} \lambda (\delta_{\max}) = 0$. Looking at (22), it is easy to see that we can upper bound $|\bar{x}(\Delta)|$ with

$$\gamma_1 (\delta_{\max}) + \gamma_e (\bar{x}_{\max} \delta_{\max})$$

where

$$\gamma_1 (\delta_{\max}) \geq \frac{1}{1 - \lambda (\delta_{\max}) \mu_{\max}} \beta_e (x'_{\max}, 0; \mu)$$

and the remaining element on the left-hand side of (22) is represented by $\gamma_e$. Thus (24) and the second limit result in (25) follows. Because $\lim_{\delta_{\max} \to 0} \lambda (\delta_{\max}) = 0$, the first limit result in (25) also follows. \hfill \square
Another corollary of (Sharon and Liberzon, 2010, Theorem 4) is as follows:

**Corollary 9.** Assume that (10) and (24) holds $\forall |x_d(\Delta)| \leq x'_m, \forall \|x_d\|_{[\Delta, \infty]} \leq \bar{x}_m, \forall \mu_k(\Delta) \leq \mu_{\max}, \forall \delta_{\max} < \delta_{\max}$ for some $x'_m$, $\bar{x}_m$, $\mu_{\max}$ and $\delta_{\max}$. Set $r'_1 = \bar{x}_m$. Assume also that for some $r_0' > 0$, $\alpha > 0$ and $\lambda < 1$,

$$\gamma_\alpha \left(1 + \alpha \right) \gamma_c \left(r_1; \delta_{\max}\right) < \lambda r,$$

and

$$\frac{1}{1 - \lambda} \gamma_c \left( |x_d(\Delta)|, 0 \right) + \frac{1}{1 - \lambda} \gamma_\alpha \left( \gamma_c \left( \delta_{\max} \right) \right) < r'_1,$$

(27)

$$\frac{1}{1 - \lambda} \gamma_\alpha \left( \delta_{\max} \right) + \frac{1}{1 - \lambda} \gamma_c \left( \gamma_\alpha \left( |x_d(\Delta)|, 0 \right); \delta_{\max} \right) < \gamma_r^{-1} \left(r'_1\right).$$

(28)

$$\frac{1}{1 - \lambda} \gamma_c \left( \delta_{\max} \right) + \frac{1}{1 - \lambda} \gamma_\alpha \left( |x_d(\Delta)|, 0 \right) + \gamma_c \left( \gamma_\alpha \left( \delta_{\max} \right) \right) < \gamma_{\max} \left(r'_1\right).$$

(29)

Then $\|x_d\|_{[\Delta, \infty]} < r'_1$ and $\|x_d\|_{[\Delta, \infty]} < \gamma_r^{-1} \left(r'_1\right)$.

We remark that having (29) imply $\|x_d\|_{[\Delta, \infty]} < \gamma_r^{-1} \left(r'_1\right)$ given (27) relies on the linearity of $\gamma_\alpha$ which was established in Lemma 3.

We are now ready to prove Theorem 1.

**Proof.** Assume $x'_m$, $\mu_{\max}$ are given. Choose $r'_1$ such that

$$r'_1 > \beta_1 \left(x'_m, 0 \right) + \gamma_\alpha \left( \sup_{\mu \in [0, \mu_{\max}]} \beta_\delta \left(x'_m, 0; \mu \right) \right).$$

(30)

We can now find $\delta_{\max} > 0$ for which (23) holds with $x'_m = r'_1$ and $\gamma_\alpha \left( \gamma_\alpha \left( |x_d(\Delta)|, 0 \right) \right) < \gamma_r^{-1} \left(r'_1\right)$.

Taking a smaller $\delta_{\max}$ if necessary, we can now also satisfy the assumptions of Corollary 9. This establishes that $x_d(\Delta) \leq x'_m$, $\bar{x}_m(\Delta) \leq x'_m$, $\mu_k(\Delta) \leq \mu_{\max}$, and $\delta_{\max} \leq \delta_{\max}$ then $\forall \gamma > \gamma_\alpha \left( |x_d(\Delta)|, 0 \right) \leq 0$ and (10) and (23) holds, as well as $\|x_d\|_{[\Delta, \infty]} < r'_1$, and $\|x_d\|_{[\Delta, \infty]} < \gamma_r^{-1} \left(r'_1\right)$.

Taking an even smaller $\delta_{\max}$ if necessary, we can make the small-gain condition between (10) and (23),

$$\gamma_\alpha \left( \gamma_\alpha \left( \gamma_c \left( \delta_{\max} \right) \right) \right) < \gamma_r^{-1} \left( r'_1 \right),$$

(31)

hold $\forall \delta_{\max} < \delta_{\max}$, so that we can apply the local Small-Gain Theorem (Corollary 7) one more time and arrive at $\forall \gamma > \gamma_\alpha \left( |x_d(\Delta)|, 0 \right)$

$$\left( x_d(t), x_d(t_0) \right) \leq \beta_2 \left( \left( x_d(t_0), \left| t - t_0 \right| \right), d \right),$$

where $\beta_2 \in C$. The last term above, $d$, is nonzero due to $d' > 0$ and $r_0' > 0$ when $\delta_{\max} > 0$. However, we can make both $d'$ and $r_1'$ arbitrarily small, and therefore also $d$, by taking a sufficiently small $\delta_{\max} > 0$. Thus we can replace $d$ with $\gamma \left( \delta_{\max} \right) \in K$.

We now bound the evolution of $x$ and $z$ from $t = 0$ to $t = \Delta$. Initially $x = 0$ and at the first quantizer by our quantizer $|z(\delta_0)| < 2 |x(0)|$, leading to $\|x\|_{[0, T_\alpha + \delta_1]} \leq \|x\|_{[0, T_\alpha + \delta_1]} \leq e^{(T_\alpha + \delta_{1}) ||z||^{2} + BK} \|x(0)\| \leq \rho_1 \|x(0)\|$. Thus

$$\|x\|_{[0, T_\alpha + \delta_1]} \leq e^{(T_\alpha + \delta_{1}) ||z||^{2} + BK} \|x(0)\| \leq \rho_1 \|x(0)\|.$$