# Stabilization of Linear Systems under Coarse Quantization and Time Delays ⋆

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**Abstract:** We consider the problem of stabilizing a control system using a coarse state quantizer in the presence of time delays. We assume the quantizer has an adjustable "center" and "zoom" parameters, and employ an alternating "zoom out"/"zoom in" mechanism in order to achieve a large region of attraction while having the system converge to a small region around the origin. This mechanism is adopted from our previous work where delays were not considered. Here we show that the control system, using the same mechanism and without making any changes in order to accommodate delays explicitly, remains stable under small delays. The main tool we use to prove the result is the nonlinear small-gain theorem.

*Keywords:* Quantization, Delays, Stability

## 1. INTRODUCTION

Networked control systems are characterized by simultaneous presence of several communication constraints, such as quantization, time scheduling, time delays, packet dropouts, interference, and so on. While early work focused on studying just one of these aspects, more recently results able to handle two or more are beginning to emerge (see, e.g., Heemels et al. (2009) and the references therein). In this paper we address two of the phenomena mentioned above, namely, state quantization and time delays.

The approach described here has its roots in two related lines of recent work. The first relevant contribution is the method for stabilizing nonlinear systems with quantization and delays presented in Liberzon (2006). The analysis in Liberzon (2006) centers around the concept of input-to-state stability (ISS) and an associated small-gain theorem, and is based on the approach of Teel (1998). An important drawback of the result given in Liberzon (2006), however, is that it does not attempt to minimize the data rate and so the bound on the number of quantization regions that it requires is very conservative. On the other hand, there have been many results on quantized stabilization with minimal data rate. In the context of nonlinear systems, an ISS control framework was developed in Liberzon and Hespanha (2005) and subsequently refined in Sharon and Liberzon (2010) to obtain ISS with respect to external disturbances. However, these results do not allow the presence of time delays.

Thus, the contribution of the present work is essentially to extend the method of Sharon and Liberzon (2010) to the case where (possibly time-varying) delays are present in addition to state quantization. Although we assume there are no external disturbances in this paper, we do rely on the ISS property which we established in Sharon and Liberzon (2010) after we show that error signals which arise due to delays can be regarded as external disturbances. The ISS small-gain analysis employed in this paper is similar in spirit to that used in Liberzon (2006), but it becomes more challenging due to the dynamics of the quantizer which are necessary to achieve minimal data rate (in Liberzon (2006) only a static quantizer was considered). We believe that, by virtue of being able to handle both quantization and delays while enforcing a minimal data rate, our result will be of greater use for analysis and design of networked control systems. In this paper we consider linear plant dynamics, but our method is nonlinear in nature and we expect it to naturally extend to suitable nonlinear systems along the lines of (Sharon and Liberzon, 2010, Section VI).

Among other noteworthy references dealing with quantization and delays, using approaches different from ours, we mention Fridman and Dambrine (2009), De Persis and Mazenc (2009), and Sailer and Wirth (2009). The first two of these papers employ Lyapunov-Krasovskii functionals for linear and nonlinear systems, respectively, while the last one handles nonlinear systems by sending time information along with the encoded state.

The outline of this paper is as follows: In §2 we define the quantized and delayed control system which we address in this paper; in §3 we recall the controller we developed in Sharon and Liberzon (2010); in §4 we present the main result of this paper; §5 is dedicated to proving the main result; concluding remarks are in §6.

## 2. PROBLEM FORMULATION

The system we consider consists of three components: the plant, the quantizer, and the controller. The continuous-time plant we are to stabilize is as follows ($t \in \mathbb{R}_{\geq 0}$):

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t) \tag{1}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the state and $\boldsymbol{u} \in \mathbb{R}^m$ is the control input.

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The quantizer samples the state of the plant every $T_s$ seconds and generates the information for the controller, $\boldsymbol{z} : \{kT_s \,|k \in \mathbb{Z}_{\geq 0}\} \to \mathbb{R}^{n_y}$:

$$\boldsymbol{z}\left(kT_s\right) = Q\left(\boldsymbol{x}\left(kT_s\right); \boldsymbol{c}\left(kT_s\right), \mu\left(kT_s\right)\right), \tag{2}$$

where $\boldsymbol{c} : \{kT_s \,|k \in \mathbb{Z}_{\geq 0}\} \to \mathbb{R}^n$ and $\mu : \{kT_s \,|k \in \mathbb{Z}_{\geq 0}\} \to \mathbb{R}_{>0}$ are quantization parameters and $Q$ is the quantization function. For convenience we will use the notation $\boldsymbol{z}_k \doteq \boldsymbol{z}\left(kT_s\right)$, and similarly for other variables.

We will present our results using the following (square) quantizer. We assume $N$ is an odd number, $N \geq 3$, which counts the number of quantization regions per state dimension. The quantizer is denoted by $\left(Q_1, \ldots, Q_n\right)^T = Q\left(\boldsymbol{x}; \boldsymbol{c}, \mu\right)$ where each scalar component is defined as follows:

$$Q_i\left(\boldsymbol{x}; \boldsymbol{c}, \mu\right) \doteq c_i + 2\mu \tag{3}$$
$$\times \begin{cases} (-N+1)/2 & x_i - c_i \leq (-N+2)\mu \\ (N-1)/2 & (N-2)\mu < x_i - c_i \\ \lceil \left(x_i - \left(c_i + \mu\right)\right) / \left(2\mu\right) \rceil & \text{otherwise.} \end{cases}$$

We will refer to $\boldsymbol{c}$ as the *center* of the quantizer, and to $\mu$ as the *zoom factor*. Note that what will actually be transferred from the quantizer to the controller will be an index to one of the quantization regions. The controller, which either generates the values $\boldsymbol{c}$ and $\mu$ and shares them with the quantizer or knows the rule by which they are generated, will use this information to convert the received index to the value of $Q$ as given in (3). This setup is the same as in Sharon and Liberzon (2010).

Due to delays, for every $k \in \mathbb{Z}_{\geq 0}$ the controller receives the information $\boldsymbol{z}_k = \boldsymbol{z}\left(kT_s\right)$ only at time $kT_s + \delta_k$ where $\delta_k \in [0, T_s)$ is the delay. The delay is unknown to the controller and it does not need to be fixed. We set $\delta_{\max} \doteq \sup_{k \geq 0} \delta_k$.

In this paper we will use the $\infty$-norm unless otherwise specified. For vectors, $|\boldsymbol{x}| \doteq |\boldsymbol{x}|_\infty \doteq \max_i |x_i|$. For continuous-time signals, $\|\boldsymbol{w}\|_{[t_1,t_2]} \doteq \max_{t \in [t_1,t_2]} |\boldsymbol{w}(t)|_\infty$, $\|\boldsymbol{w}\| \doteq \|\boldsymbol{w}\|_{[0,\infty)}$. For discrete-time signals, $\|\boldsymbol{z}\|_{\{k_1 \ldots k_2\}} \doteq \max_{k \in \{k_1 \ldots k_2\}} |\boldsymbol{z}_k|_\infty$, $\|\boldsymbol{z}\| \doteq \|\boldsymbol{z}\|_{\{0 \ldots \infty\}}$. For matrices we use the induced norm corresponding to the specified norm ($\infty$-norm if none specified). For piecewise continuous signals we will use the superscripts $+$ and $-$ to denote the right and left continuous limits, respectively: $\boldsymbol{x}_k^+ \doteq \boldsymbol{x}^+\left(kT_s\right) \doteq \lim_{t \searrow 0} \boldsymbol{x}\left(kT_s + t\right)$, $\boldsymbol{x}_k^- \doteq \boldsymbol{x}^-\left(kT_s\right) \doteq \lim_{t \nearrow 0} \boldsymbol{x}\left(kT_s + t\right)$.

## 3. CONTROLLER DESIGN

We implement the same controller as in Sharon and Liberzon (2010). One of the tasks of the controller is to generate the state estimate, $\hat{\boldsymbol{x}}\left(t\right)$, for which we will use the notation $\hat{\boldsymbol{x}}_k \doteq \hat{\boldsymbol{x}}\left(kT_s + \delta_k\right)$. The controller keeps and updates a discrete time step variable, $k \in \mathbb{N}$, whose value will correspond to the current sampling time of the continuous system. When a new measurement is produced at times $kT_s$, it may be used to update the state estimate $\hat{\boldsymbol{x}}_k$ where $k$ is the discrete time step. At each discrete time step, the controller will operate in one of three modes: *capture*, *measurement update* or *escape detection*. The current mode will be stored in the variable $\mathit{mode}(k) \in \{\mathbf{capture}, \mathbf{update}, \mathbf{detect}\}$. The controller will also use $p_k \in \mathbb{Z}$ and $\mathit{saturated}(k) \in \{\mathbf{true}, \mathbf{false}\}$ as auxiliary variables. The variable $p_k$ counts the number of sampling times at which the controller was in the *measurement update* mode since the last sampling time at which it was either in the *capture* mode or the *escape detection* mode. We note that the difference between the *measurement update* mode and the *escape detection* mode is that in the former we set the quantizer so as to minimize the estimation error, but this comes at the expense of not being able to detect saturation.

We assume the control system is activated at $k = 0$ ($t = 0$). We initialize $\hat{\boldsymbol{x}}\left(0\right) = \mathbf{0}$, $\mathit{mode}(0) = \mathit{capture}$, $p_0 = 0$, and $\mu_0 = s$, where $s$ is a positive constant which will be regarded as a design parameter. We also use the following design parameters: $\alpha \in \mathbb{R}_{>0}$, $\Omega_{\text{out}} \in \mathbb{R}$ such that $\Omega_{\text{out}} > \left\|e^{T_s A}\right\|$, and $P \in \mathbb{Z}$ such that $P \geq 1$. We refer the reader to (Sharon and Liberzon, 2007, §V) for a detailed qualitative discussion on how each design parameter affects the system performance. The last design parameter is the static feedback control law, $K$, which should be chosen such that $A + BK$ is Hurwitz.

On the time interval between the arrivals of new measurements, $t \in [kT_s + \delta_k, (k+1)\,T_s + \delta_{k+1}]$, the controller continuously updates the state estimate and the control input based on the nominal system dynamics:

$$\dot{\hat{\boldsymbol{x}}}(t) = A\hat{\boldsymbol{x}}(t) + B\boldsymbol{u}\left(t\right) \qquad \boldsymbol{u}\left(t\right) = K\hat{\boldsymbol{x}}\left(t\right). \tag{4}$$

Whenever a new measurement is received from the quantizer at time $kT_s + \delta_k$, the controller executes sequentially Algorithm 1–Algorithm 5:

**Algorithm 1** Preliminaries

```
if ∃i such that (z_k)_i = (c_k)_i ± (N − 1)μ_k then
    set saturated(k) = true
else
    set saturated(k) = false
end if
mode(k + 1) = mode(k)
```

**Algorithm 2** *capture* mode

```
if mode(k) = capture then
    set p_k = 0
    if not saturated(k) then
        update the state estimate: x̂(kT_s + δ_k) = z_k
        set mode(k + 1) = update
    end if
end if
```

**Algorithm 3** *measurement update* mode

```
if mode(k) = update then
    set p_k = p_{k−1} + 1
    update the state estimate: x̂(kT_s + δ_k) = z_k
    if p_k = P − 1 then
        set mode(k + 1) = detect
    end if
end if
```

## 4. MAIN RESULT

Let $\mu_0' = 1$, $\mu_k = \left(\left\|e^{T_s A}\right\| \mu_{k-1}' + \alpha\right)/N$, $k = 1, \ldots, P-1$, $\mu_P' = \left(\left\|e^{T_s A}\right\| \mu_{P-1}' + \alpha\right)/\left(N-2\right)$. If $\mu_P' < 1$ then we say that the design parameter $\alpha$ satisfies the *convergence property*. In (Sharon and Liberzon, 2010, Lemma 1) we

**Algorithm 4** *escape detection* mode

**if** *mode*$(k)$ = **detect then**
  **if not** *saturated*$(k)$ **then**
    update the state estimate: $\hat{\boldsymbol{x}}(kT_s + \delta_k) = \boldsymbol{z}_k$
    set $p_k = 0$
    set *mode*$(k+1)$ = **update**
  **end if**
**else**
  set $p_k = 0$ and $\mu_k = s$
  switch to *capture* mode: set *mode*$(k+1)$ = **capture**
**end if**

**Algorithm 5** preparing for next sampling

**if** *mode*$(k+1)$ = **capture then**
  set $\mu_{k+1} = \Omega_{out}\mu_k$
**else if** *mode*$(k+1)$ = **update then**
  set $\mu_{k+1} = (\|A_d\|\mu_k + \alpha\|\mu_{k-p_k}\|)(N)$
**else if** *mode*$(k)$ = **detect then**
  set $\mu_{k+1} = (\|A_d\|\mu_k + \alpha\|\mu_{k-p_k}\|)(N-2)$
**end if**
set $\boldsymbol{c}_{k+1} = \exp(T_s(A+BK))\hat{\boldsymbol{x}}(kT_s + \delta_k)$

proved that a necessary and sufficient condition for the existence of such an $\alpha$ is $\left\|e^{T_s A}\right\|/N < 1$.

*Theorem 1.* Given an implementation of the controller above with any valid choice for the design parameters such that $\alpha$ satisfies the convergence property, the closed loop system will have the following semiglobal stability property: For every $x_{\max} \geq 0$, there exists a sufficiently small but strictly positive $\bar{\bar{\delta}}_{\max}$ such that if $\delta_{\max} \leq \bar{\bar{\delta}}_{\max}$ then the following bound, $\forall t \geq 0$:

$$|\boldsymbol{x}(t)| \leq \beta(|\boldsymbol{x}(0)|, t) + \gamma(\delta_{\max}) \tag{5}$$

holds whenever $|\boldsymbol{x}(0)| \leq x_{\max}$, where the function $\beta$ is of class $\mathcal{KL}$ [1] ($\beta \in \mathcal{KL}$) and $\gamma$ is of class $\mathcal{K}$ ($\gamma \in \mathcal{K}$).

*Remark*: Known results on delays, Liberzon (2006) for example, provide what can be interpreted as a more general result than (5), in which the time 0 is replaced with $t_0$ and the bound holds for arbitrary $t_0$. In fact, an intermediate step in proving Theorem 1 (see (31) below) does provide a similar result which holds for arbitrary $t_0$. However, results for systems with delays which hold for arbitrary $t_0$ require to know a history of the state over some nonzero time interval. By constraining ourselves to $t_0 = 0$ we are able to get a bound which only depends on the state at this time instance.

## 5. PROOF

We start with a brief overview of the proof. In addition to the state signal, $\boldsymbol{x}(t)$, we define a state estimation error signal, $\tilde{\boldsymbol{x}}(t) = \hat{\boldsymbol{x}}(t) - \boldsymbol{x}(t-\delta)$ (the explicit dependence of $\delta$ on $t$ will be provided in the proof itself). We also define two additional signals, $\boldsymbol{\theta}_x(t) = \boldsymbol{x}(t-\delta) - \boldsymbol{x}(t)$ and $\boldsymbol{\theta}_e(t) = \tilde{\boldsymbol{x}}(t-\delta) - \tilde{\boldsymbol{x}}(t)$. We use a small-gain argument between $\boldsymbol{x}$ and $\boldsymbol{\theta}_x$ in Lemma 3 to show that for a sufficiently small delay, there exists an ISS relation between the state estimation error signal (as the only input) and the state signal. We establish that the two signals $\boldsymbol{\theta}_x(t)$ and $\boldsymbol{\theta}_e(t)$ enter the system as external disturbances, and recall in Corollary 4 our previous result that the state estimation error signal posses the ISS property with respect to external disturbances. We then use a small-gain argument between $\tilde{\boldsymbol{x}}$ and $\boldsymbol{\theta}_e$ in Lemma 8 to show that for a sufficiently small delay, there exists a local ISS relation between the state signal (as the only input) and the state estimation error signal. Finally, in the proof of Theorem 1 we use another small-gain argument between these two established ISS relations to derive the desired result.

We define two additional classes of functions. We say that a function $\beta(\nu, t; \mu)$ if of class $\overline{\mathcal{KL}}$ when as a function of its first two arguments with the third argument fixed, it is of class $\mathcal{KL}$, and it is a continuous function of its third argument when the first two arguments are fixed. We say that a function $\gamma(\nu; \mu)$ if of class $\overline{\mathcal{K}}_\infty$ when as a function of its first argument with the second argument fixed, it is of class $\mathcal{K}_\infty$, and it is a continuous function of its second argument when the first argument is fixed. We adopt the following notation from Teel (1998): $\boldsymbol{x}_d(t) \doteq \|\boldsymbol{x}\|_{[t-\Delta,t]}$ and $\tilde{\boldsymbol{x}}_d(t) \doteq \|\tilde{\boldsymbol{x}}\|_{[t-\Delta,t]}$ where

$$\Delta \doteq 2T_s + \delta_{\max}.$$

We start with a technical lemma:

*Lemma 2.* Let a system with state $\boldsymbol{x}$ satisfy the following relation, $\forall t \geq t_0 \geq \Delta$:

$$|\boldsymbol{x}(t)| \leq \beta_x(|\boldsymbol{x}(t_0)|, t - t_0) + \gamma_x\Big(\|\boldsymbol{x}_d\|_{[t_0,t]}\Big) + \gamma_w\Big(\|\boldsymbol{w}\|_{[t_0,t]}\Big) \tag{6}$$

where $\beta_x \in \mathcal{KL}$, and $\gamma_x, \gamma_w \in \mathcal{K}_\infty$. If $\gamma_x(r) < \lambda r$ for some $\lambda < 1$ then for every function $\gamma \in \mathcal{K}_\infty$ such that

$$\gamma(\nu) \geq \left(1 + \sqrt{\frac{\lambda}{1-\lambda}}\right)\left(1 + \lambda\left(1 + \sqrt{\frac{\lambda}{1-\lambda}}\right)\right)\gamma_w(\nu) \tag{7}$$

there exists a function $\beta \in \mathcal{KL}$ such that $\forall t \geq t_0 \geq \Delta$:

$$|\boldsymbol{x}_d(t)| \leq \beta(|\boldsymbol{x}_d(t_0)|, t - t_0) + \gamma\left(\|\boldsymbol{w}\|_{[t_0,t]}\right). \tag{8}$$

**Proof.** First we have $\forall t \geq t_0 \geq \Delta$:

$$|\boldsymbol{x}_d(t)| \leq \beta_d(|\boldsymbol{x}_d(t_0)|, t - t_0) + \gamma_d\left(\|\boldsymbol{x}\|_{[t_0,t]}\right)$$

where $\beta_d(\nu, t) = 1_{t<\Delta}\nu + 1_{t\geq\Delta}e^{-1/\epsilon(t-\Delta)}$ with arbitrary $\epsilon > 0$ ($1_{t<\Delta}$ is the characteristic function whose value is 1 if $t < \Delta$ and 0 otherwise) and $\gamma_d(\nu) = \nu$. Note that $\beta_d \in \mathcal{KL}$ and $\gamma_d \in \mathcal{K}_\infty$. Defining $y(t) \doteq \gamma_x(|\boldsymbol{x}_d(t)|)$ we can have $\forall t \geq t_0 \geq \Delta$:

$$|y(t)| \leq \beta_y(|y(t_0)|, t - t_0) + \gamma_y\left(\|\boldsymbol{x}\|_{[t_0,t]}\right)$$

where $\beta_y(\nu, t) = \gamma_x\left(\beta_d\left(\gamma_x^{-1}(\nu), t\right)\right) \in \mathcal{KL}$ and $\gamma_y(\nu) = \gamma_x(\nu) \in \mathcal{K}_\infty$.

Invoking the Small-Gain Theorem (Jiang et al., 1994, Theorem 2.1), with $\beta_1(\nu, t) = \beta_x(\nu, t)$, $\gamma_1^y(\nu) = \nu$, $\gamma_1^u(\nu) = \gamma_w(\nu)$, $\beta_2(\nu, t) = \beta_d(\nu, t)$, $\gamma_2^y(\nu) = \gamma_x(\nu)$, $\gamma_2^u(\nu) = 0$, and $\rho_1 = \rho_2 = 1/\sqrt{\lambda} - 1$, we can get functions $\beta', \beta'' \in \mathcal{KL}$ such that $\forall t \geq t_0 \geq \Delta$:

[1] A function $\alpha : [0,\infty) \to [0,\infty)$ is said to be of class $\mathcal{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$. A function $\alpha : [0,\infty) \to [0,\infty)$ is said to be of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and also unbounded. A function $\beta : [0,\infty) \times [0,\infty) \to [0,\infty)$ is said to be of class $\mathcal{KL}$ if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each fixed $t \geq 0$ and $\beta(s,t)$ decreases to 0 as $t \to \infty$ for each fixed $s \geq 0$.

$$
\begin{aligned}
|\boldsymbol{x}(t)| \leq & \beta'\left(\left|\begin{array}{c}\boldsymbol{x}(t_0)\\ \gamma_x(|\boldsymbol{x}_d(t_0)|)\end{array}\right|, t-t_0\right)+\gamma\left(\|\boldsymbol{w}\|_{[t_0,t]}\right)\\
\leq & \beta''\left(|\boldsymbol{x}_d(t_0)|, t-t_0\right)+\gamma\left(\|\boldsymbol{w}\|_{[t_0,t]}\right)
\end{aligned}
$$

for every $\gamma \in \mathcal{K}_\infty$ which satisfies (7). Because it must hold that $\beta''(\nu, 0) \geq \nu$ we can arrive at (8) with $\beta(\nu, t) = \beta''(\nu, \max\{0, t-\Delta\})$. □

Define $k(t) \doteq \max\{k \in \mathbb{Z}_{\geq 0} \,|kT_s + \delta_k \leq t\}$, the index of the last sampling which arrived at the controller before time $t$. With this definition we can write:

$$
\begin{aligned}
\dot{\boldsymbol{x}}(t) &= A\boldsymbol{x}(t) + BK\left(\boldsymbol{x}\left(t-\delta_{k(t)}\right)+\tilde{\boldsymbol{x}}(t)\right)\\
&= (A+BK)\,\boldsymbol{x}(t) + BK\left(\boldsymbol{\theta}_x(t)+\tilde{\boldsymbol{x}}(t)\right) \qquad (9)
\end{aligned}
$$

where $\boldsymbol{\theta}_x(t) \doteq \boldsymbol{x}\left(t-\delta_{k(t)}\right) - \boldsymbol{x}(t)$ and $\tilde{\boldsymbol{x}}(t) \doteq \hat{\boldsymbol{x}}(t) - \boldsymbol{x}\left(t-\delta_{k(t)}\right)$.

*Lemma 3.* There exists a sufficiently small, but strictly positive, $\bar{\delta}_{\max}$, such that if $\delta_{\max} \leq \bar{\delta}_{\max}$ then the following ISS relation, $\forall t \geq t_0 \geq \Delta$:

$$
|\boldsymbol{x}_d(t)| \leq \beta_x\left(|\boldsymbol{x}_d(t_0)|, t-t_0\right) + \gamma_x\left(\|\tilde{\boldsymbol{x}}_d\|_{[t_0,t]}\right) \qquad (10)
$$

holds where $\beta_x \in \mathcal{KL}$ and $\gamma_x \in \mathcal{K}_\infty$ is a linear function.

**Proof.** A standard result on ISS for linear systems is that the system defined by (9) follows

$$
|\boldsymbol{x}(t)| \leq \tilde{\beta}_x\left(|\boldsymbol{x}(t_0)|, t-t_0\right) + \tilde{\gamma}_x\Big(\|\boldsymbol{\theta}_x\|_{[t_0,t]}\Big) + \tilde{\gamma}_x\Big(\|\tilde{\boldsymbol{x}}\|_{[t_0,t]}\Big) \qquad (11)
$$

where $\tilde{\beta}_x \in \mathcal{KL}$ and $\tilde{\gamma}_x \in \mathcal{K}_\infty$ is a linear function. For example one can take $\tilde{\beta}_x(\nu, t) = ce^{-\sigma t}\nu$ and $\tilde{\gamma}_x(\nu) = \frac{c\|BK\|}{\sigma}\nu$ where $c > 0$ and $\sigma > 0$ are such that $\left\|e^{(A+BK)t}\right\| \leq ce^{-\sigma t}$ $\forall t \geq 0$.

We also have from the first line in (9), $\forall t \geq \Delta$:

$$
\begin{aligned}
|\boldsymbol{\theta}_x(t)| =& \left|-\int_{t-\delta_{k(t)}}^{t} A\boldsymbol{x}(\tau) + BK\boldsymbol{x}\left(\tau-\delta_{k(\tau)}\right) + BK\tilde{\boldsymbol{x}}(\tau)\,d\tau\right|\\
\leq& \delta_{max}\left(\|A\| + \|BK\|\right)\|\boldsymbol{x}\|_{\left[t-\delta_{k(t)}-\delta_{k\left(t-\delta_{k(t)}\right)},t\right]} +\\
&\delta_{max}\|BK\|\,\|\tilde{\boldsymbol{x}}\|_{\left[t-\delta_{k(t)},t\right]}\\
\leq& \delta_{max}\left(\|A\| + \|BK\|\right)|\boldsymbol{x}_d(t)| +\\
&\delta_{max}\|BK\|\,|\tilde{\boldsymbol{x}}_d(t)|\,. \qquad (12)
\end{aligned}
$$

For the last inequality we used the fact that $\Delta \geq 2\delta_{\max}$. Substituting this into (11), we get (6) with

$$
\begin{aligned}
\gamma_x(\nu) &= \tilde{\gamma}_x\left(\delta_{max}\left(\|A\| + \|BK\|\right)\nu\right)\\
\gamma_w(\nu) &= \tilde{\gamma}_x\left(\delta_{max}\|BK\|\nu\right) + \tilde{\gamma}_x(\nu)
\end{aligned}
$$

(we used the fact that $\tilde{\gamma}_x$ is a linear function). Choosing $\bar{\delta}_{\max}$ such that $\tilde{\gamma}_x\left(\bar{\delta}_{\max}\left(\|A\| + \|BK\|\right)\nu\right) \leq \nu$ $\forall \nu$, (10) follows by Lemma 2. □

Define $\bar{k}(t) = \lfloor t/T_s \rfloor$. Another way to expand (9) is as follows, $\forall t \geq \delta_0$:

$$
\begin{aligned}
\dot{\boldsymbol{x}}(t) &= A\boldsymbol{x}(t) + B\boldsymbol{u}(t) = A\boldsymbol{x}(t) + BK\hat{\boldsymbol{x}}(t)\\
&= A\boldsymbol{x}(t) + BK\left(\hat{\boldsymbol{x}}\left(t+\delta_{\bar{k}(t)}\right) + \hat{\boldsymbol{x}}(t) - \hat{\boldsymbol{x}}\left(t+\delta_{\bar{k}(t)}\right)\right)\\
&= A\boldsymbol{x}(t) + B\boldsymbol{u}\Big(t+\delta_{\bar{k}(t)}\Big) + BK\Big(\boldsymbol{\theta}_e\Big(t+\delta_{\bar{k}(t)}\Big) + \boldsymbol{\theta}_x(t)\Big)
\end{aligned}
\qquad (13)
$$

where $\boldsymbol{\theta}_e(t) \doteq \tilde{\boldsymbol{x}}\left(t-\delta_{k(t)}\right) - \tilde{\boldsymbol{x}}(t)$. For $t \geq T_s + \delta_1$, $t \neq kT_s + \delta_k$ $\forall k$, the state estimate evolves according to (4) and thus the estimation error, $\tilde{\boldsymbol{x}}$, evolves according to

$$
\begin{aligned}
\dot{\tilde{\boldsymbol{x}}}(t) =& \dot{\hat{\boldsymbol{x}}}(t) - \dot{\boldsymbol{x}}\left(t-\delta_{k(t)}\right)\\
=& A\tilde{\boldsymbol{x}}(t) - BK\left(\boldsymbol{\theta}_e(t) + \boldsymbol{\theta}_x\left(t-\delta_{k(t)}\right)\right). \qquad (14)
\end{aligned}
$$

Denoting

$$
\begin{aligned}
\boldsymbol{w}(t) \doteq& -BK\left(\boldsymbol{\theta}_e(t) + \boldsymbol{\theta}_x\left(t-\delta_{k(t)}\right)\right),\\
\boldsymbol{w}_k^d \doteq& \int_{kT_s+\delta_k}^{(k+1)T_s+\delta_k} e^{A(k+1)T_s+\delta_k-t}\boldsymbol{w}(t)\,dt,
\end{aligned}
$$

we have that $\forall k \geq 1$:

$$
\begin{aligned}
\boldsymbol{c}_{k+1} - \boldsymbol{x}_{k+1} &= \boldsymbol{c}\left((k+1)\,T_s\right) - \boldsymbol{x}\left((k+1)\,T_s\right)\\
&= e^{T_sA}\tilde{\boldsymbol{x}}\left(kT_s+\delta_k\right) + \boldsymbol{w}_k^d \doteq e^{T_sA}\tilde{\boldsymbol{x}}_k + \boldsymbol{w}_k^d \qquad (15)
\end{aligned}
$$

where $\boldsymbol{c}$ is the quantization parameter defining the center of the quantizer. In (Sharon and Liberzon, 2010, Proposition 2) we proved that if the system satisfies (15), then the following holds:

*Corollary 4.* There exist functions $\beta_{e,d} \in \overline{\mathcal{KL}}$ and $\gamma_{e,d} \in \overline{\mathcal{K}}_\infty$ such that $\forall k \geq k_0 \geq 1$:

$$
\begin{aligned}
|\tilde{\boldsymbol{x}}_k| \leq& \beta_{e,d}\left(|\tilde{\boldsymbol{x}}_{k_0}|, k-k_0; \mu_{k_0}\right) + \gamma_{e,d}\left(\left\|\boldsymbol{w}^d\right\|_{\{k_0,k-1\}}; \mu_{k_0}\right)\\
\mu_k \leq& \psi\left(\|\tilde{\boldsymbol{x}}\|_{\{k_0,k-1\}}; \mu_{k_0}\right). \qquad (16)
\end{aligned}
$$

The function $\psi(\cdot,\cdot)$ as a function of its first argument when its second argument is fixed, is continuous, non-decreasing and non-negative. As a function of its second argument when its first argument is fixed, it is continuous.

*Lemma 5.* The delayed estimation error, $\tilde{\boldsymbol{x}}_d$, satisfies the following relation, $\forall t \geq t_0 \geq \Delta$:

$$
\begin{aligned}
|\tilde{\boldsymbol{x}}_d(t)| \leq& \tilde{\beta}_e\left(|\tilde{\boldsymbol{x}}_d(t_0)|, t-t_0; \mu_{k(t_0)}\right) +\\
&\tilde{\gamma}_e\left(\delta_{\max}\|\tilde{\boldsymbol{x}}_d\|_{[t_0,t]}; \mu_{k(t_0)}\right) +\\
&\tilde{\gamma}_w\left(\delta_{\max}\|\boldsymbol{x}_d\|_{[t_0,t]}; \mu_{k(t_0)}\right). \qquad (17)
\end{aligned}
$$

where $\tilde{\beta}_e \in \overline{\mathcal{KL}}$ and $\tilde{\gamma}_e, \tilde{\gamma}_w \in \overline{\mathcal{K}}_\infty$.

**Proof.** We can bound $\boldsymbol{w}_k^d$, $\forall k \geq 1$, as

$$
\begin{aligned}
\left|\boldsymbol{w}_k^d\right| \leq& e^{T_s\|A\|}\|BK\|\int_{kT_s+\delta_k}^{(k+1)T_s+\delta_k}|\boldsymbol{\theta}_e(t)|\,dt +\\
&e^{T_s\|A\|}\|BK\|\,T_s\|\boldsymbol{\theta}_x\|_{[kT_s,(k+1)T_s]}\,.
\end{aligned}
$$

We can also bound the estimation error between updates, $\forall k \geq 1$ and $\forall t \in [kT_s + \delta_k, (k+1)\,T_s + \delta_{k+1}]$:

$$
\begin{aligned}
|\tilde{\boldsymbol{x}}(t)| \leq& e^{(T_s+\delta_{\max})\|A\|}|\tilde{\boldsymbol{x}}_k| + e^{(T_s+\delta_{\max})\|A\|}\int_{kT_s+\delta_k}^{t}|\boldsymbol{w}(\tau)|\,d\tau\\
\leq& e^{(T_s+\delta_{\max})\|A\|}|\tilde{\boldsymbol{x}}_k| + e^{(T_s+\delta_{\max})\|A\|}\|BK\| \times\\
&\left(\int_{kT_s+\delta_k}^{t}|\boldsymbol{\theta}_e(\tau)|\,d\tau + (T_s+\delta_{\max})\|\boldsymbol{\theta}_x\|_{[kT_s,t-\delta_k]}\right).
\end{aligned}
$$

Combining these two bounds with (16) and the first inequality in (12), we can arrive at, $\forall t \geq \Delta$:

$$
\begin{aligned}
|\tilde{\boldsymbol{x}}(t)| \leq& \beta_{e,e}\left(\left|\tilde{\boldsymbol{x}}_{k(t_0)}\right|, k(t)\,T_s - k(t_0)\,T_s; \mu_{k(t_0)}\right) +\\
&\gamma_{e,\theta}\left(\max_{k\in[k(t_0),k(t)]}\int_{kT_s+\delta_k}^{\min\{(k+1)T_s+\delta_{\max},t\}}|\boldsymbol{\theta}_e(\tau)|\,d\tau; \mu_{k(t_0)}\right) +\\
&\gamma_{e,e}\left(\delta_{\max}\|\tilde{\boldsymbol{x}}\|_{[k(t_0)T_s-\delta_{\max},t]}; \mu_{k(t_0)}\right) +\\
&\gamma_{e,x}\left(\delta_{\max}\|\boldsymbol{x}\|_{[k(t_0)T_s-2\delta_{\max},t]}; \mu_{k(t_0)}\right). \qquad (18)
\end{aligned}
$$

where $\beta_{e,e} \in \overline{\mathcal{KL}}$ and $\gamma_{e,\theta}, \gamma_{e,x}, \gamma_{e,e} \in \overline{\mathcal{K}}_\infty$.

From the definition of $\boldsymbol{\theta}_e$, $\forall t \geq \min\{2\delta_0, T_s + \delta_1\}$:

$$\boldsymbol{\theta}_e(t) = -\int_{t-\delta_{k(t)}}^{t} \dot{\tilde{\boldsymbol{x}}}(\tau)\, d\tau - \sum_{\tau \in \left(t-\delta_{k(t)}, t\right] \cap \chi} \left(\tilde{\boldsymbol{x}}(\tau) - \tilde{\boldsymbol{x}}^-(\tau)\right) \quad (19)$$

where $\chi \doteq \{t \geq 0 \,|\, \exists k \in \mathbb{N}$ such that $\tau = kT_s + \delta_k\}$. Each $t \in \chi$ affects $\boldsymbol{\theta}_e$ through the second term in (19) only in a time interval of length at most $\delta_{\max}$. The set $\left(kT_s + \delta_k - \delta_{k(kT_s+\delta_k)}, (k+1)T_s + \max\{\delta_k, \delta_{k+1}\}\right] \cap \chi$ contains at most two element $\forall k \geq 1$. Using also (14), we can finally arrive at the bound: $\forall k \geq 2$ and $\forall t \in [kT_s + \delta_k, \max\{(k+1)T_s + \delta_k, (k+1)T_s + \delta_{k+1}\})$:

$$\begin{aligned}\int_{kT_s+\delta_k}^{t} |\boldsymbol{\theta}_e(\tau)|\, d\tau \leq\; & 4\delta_{\max} \|\tilde{\boldsymbol{x}}\|_{[kT_s,t]} + \\ & \delta_{\max}(T_s + \delta_{\max})(\|A - BK\| + \|BK\|)\|\tilde{\boldsymbol{x}}\|_{[kT_s-\delta_{k-1},t]} + \\ & \delta_{\max}(T_s + \delta_{\max})\, 2\|BK\|\, \|\boldsymbol{x}\|_{[kT_s-\delta_{k-1}-\delta(kT_s-\delta_{k-1}),t-\delta_k]}.\end{aligned} \quad (20)$$

Using (20) in (18) and the same argument we used at the end of the proof of Lemma 2 to move from a bound on $|\boldsymbol{x}(t)|$ to a bound on $|\boldsymbol{x}_d(t)|$, we can arrive at the result stated in the lemma. □

A corollary of (Sharon and Liberzon, 2010, Theorem 4) gives us the following:

*Corollary 6.* Assume that (17) holds, and that there exist $r_1 > r_0 \geq 0$, and $\lambda < 1$ such that $\forall r \in [r_0, r_1]$:

$$\tilde{\gamma}_e\left(\delta_{\max} r; \mu_{k(\Delta)}\right) \leq \lambda r \quad (21)$$

and

$$\frac{1}{1-\lambda}\Big(\tilde{\beta}_e\left(|\tilde{\boldsymbol{x}}_d(\Delta)|, 0; \mu_{k(\Delta)}\right) + \tilde{\gamma}_w\left(\delta_{\max}\|\boldsymbol{x}_d\|_{[\Delta,\infty]}; \mu_{k(\Delta)}\right)\Big) < r_1. \quad (22)$$

Then $\|\tilde{\boldsymbol{x}}_d\|_{[\Delta,\infty]} < r_1$.

A corollary of the Small-Gain Theorem (Jiang et al., 1994, Theorem 2.1) gives us the following local result:

*Corollary 7.* Let $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{w}$ be three signals satisfying $\forall t \geq t_0 \geq 0$

$$\begin{aligned}|\boldsymbol{x}_1(t)| \leq& \beta_1(|\boldsymbol{x}_1(t_0)|, t - t_0) + \gamma_{1,x}\left(\|\boldsymbol{x}_2\|_{[t_0,t]}\right) + \\ & \gamma_{1,w}\left(\|\boldsymbol{w}\|_{[t_0,t]}\right) + d_1 \\ |\boldsymbol{x}(t)| \leq& \beta_1(|\boldsymbol{x}_2(t_0)|, t - t_0) + \gamma_{2,x}\left(\|\boldsymbol{x}_1\|_{[t_0,t]}\right) + \\ & \gamma_{2,w}\left(\|\boldsymbol{w}\|_{[t_0,t]}\right) + d_2\end{aligned}$$

where $\beta_1, \beta_2 \in \mathcal{KL}$, $\gamma_{1,x}, \gamma_{1,w}, \gamma_{2,x}, \gamma_{2,w} \in \mathcal{K}$ and $d_1 \geq 0$, $d_2 \geq 0$. Assume that for some $r_1 > r_0 \geq 0$ the small-gain condition

$$\gamma_{1,x}(\gamma_{2,x}(r)) < r, \qquad \forall r \in [r_0, r_1]$$

holds and it can be guaranteed that

$$\|\boldsymbol{x}_1\|_{[0,\infty]} < r_1, \qquad \|\boldsymbol{x}_2\|_{[0,\infty]} < \gamma_{1,x}^{-1}(r_1).$$

Then we can get that $\forall t \geq t_0 \geq 0$:

$$\begin{aligned}\left|\begin{matrix}\boldsymbol{x}_1(t) \\ \boldsymbol{x}_2(t)\end{matrix}\right| \leq& \beta\left(\left|\begin{matrix}\boldsymbol{x}_1(t_0) \\ \boldsymbol{x}_2(t_0)\end{matrix}\right|, t - t_0\right) + \gamma\left(\gamma_{1,w}\left(\|\boldsymbol{w}\|_{[t_0,t]}\right)\right) + \\ & \gamma\left(\gamma_{2,w}\left(\|\boldsymbol{w}\|_{[t_0,t]}\right)\right) + d\end{aligned}$$

where $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$. Furthermore, in the limit as $d_1 \to 0$, $d_2 \to 0$, $r_0 \to 0$, we get $d = 0$.

With these two corollaries we derive the following lemma:

*Lemma 8.* For any $d' > 0$, $x'_{\max}$, $\bar{x}_{\max}$ and $\mu_{\max}$ there exists a sufficiently small, but strictly positive, $\bar{\delta}_{\max}$, such that if $\delta_{\max} \leq \bar{\delta}_{\max}$ then the following ISS relation, $\forall t \geq t_0 \geq \Delta$:

$$|\tilde{\boldsymbol{x}}_d(t)| \leq \beta_e(|\tilde{\boldsymbol{x}}_d(t_0)|, t - t_0) + \gamma_e\left(\delta_{\max}\|\boldsymbol{x}_d\|_{[t_0,t]}\right) + d' \quad (23)$$

where $\beta_e \in \mathcal{KL}$ and $\gamma_e \in \mathcal{K}$ holds for all $\forall |\tilde{\boldsymbol{x}}_d(\Delta)| \leq x'_{\max}$, $\forall \|\boldsymbol{x}_d\|_{[\Delta,\infty]} \leq \bar{x}_{\max}$, and $\forall \mu_{k(\Delta)} \leq \mu_{\max}$. Furthermore, for $\forall \delta_{\max} \leq \bar{\delta}_{\max}$, we can write

$$\|\tilde{\boldsymbol{x}}_d\|_{[\Delta,\infty]} \leq \bar{\gamma}_1(\delta_{\max}) + \bar{\gamma}_e(\bar{x}_{\max}; \delta_{\max}) \quad (24)$$

where

$$\begin{aligned}\lim_{\delta_{\max}\searrow 0} \bar{\gamma}_1(\delta_{\max}) &= \sup_{\mu \in [0,\mu_{\max}]} \tilde{\beta}_e(x'_{\max}, 0; \mu) \\ \lim_{\delta_{\max}\searrow 0} \bar{\gamma}_e(\bar{x}_{\max}; \delta_{\max}) &= 0\end{aligned} \quad (25)$$

**Proof.** We first note that for any $x'_{\max} \geq 0$, $\bar{x}_{\max} \geq 0$, $\mu_{\max} \geq 0$, $r_1 > \max_{\mu \in [0,\mu_{\max}]} \tilde{\beta}_e(x'_{\max}, 0; \mu)$, and $r_0 \in (0, r_1)$, one can find $\bar{\delta}_{\max} > 0$ and $\lambda < 1$ for which the assumptions in Corollary 6 are satisfied $\forall |\tilde{\boldsymbol{x}}_d(\Delta)| \leq x'_{\max}$, $\forall \|\boldsymbol{x}_d\|_{[\Delta,\infty]} \leq \bar{x}_{\max}$, and $\forall \mu_{k(\Delta)} \leq \mu_{\max}$ if $\delta_{\max} \leq \bar{\delta}_{\max}$. We remark that because $\tilde{\gamma}_e(r, \mu)$, for any fixed $\mu$, grows faster than any linear function of $r$ both at $r = 0$ and $r = \infty$, one cannot choose $r_0 = 0$ or $r_1 = \infty$ and still satisfy the assumptions in the Corollary.

Once the assumptions of Corollary 6 are satisfied, we can replace $\mu_{k(t_0)}$ in (17) with $\bar{\mu} = \max_{\mu \in [0,\mu_{\max}]} \psi(r_1; \mu)$ and write $\forall t \geq t_0 \geq \Delta$:

$$\begin{aligned}|\tilde{\boldsymbol{x}}_d(t)| \leq& \tilde{\beta}'_e(|\tilde{\boldsymbol{x}}_d(t_0)|, t - t_0) + \tilde{\gamma}'_e\left(\delta_{\max}\|\tilde{\boldsymbol{x}}_d\|_{[t_0,t]}\right) + \\ & \tilde{\gamma}'_w\left(\delta_{\max}\|\boldsymbol{x}_d\|_{[t_0,t]}\right), \\ \|\tilde{\boldsymbol{x}}_d\|_{[\Delta,\infty]} <& r_1\end{aligned} \quad (26)$$

$\forall |\tilde{\boldsymbol{x}}_d(\Delta)| \leq x'_{\max}$, $\forall \|\boldsymbol{x}_d\|_{[\Delta,\infty]} \leq \bar{x}_{\max}$, $\forall \mu_{k(\Delta)} \leq \mu_{\max}$ and $\forall \delta_{\max} < \bar{\delta}_{\max}$ where $\tilde{\beta}'_e \in \mathcal{KL}$ and $\tilde{\gamma}'_e, \tilde{\gamma}'_w \in \mathcal{K}$. Taking $\bar{\delta}_{\max}$ to be smaller if necessary, we can also have

$$\tilde{\gamma}'_e(\delta_{\max} r) < r \qquad \forall r \in [r_0, r_1].$$

We can now use the local version of the Small-Gain Theorem (Corollary 7), similarly to how we used the Small-Gain Theorem in Lemma 2, and arrive at (23).

Note that when applying Corollary 7 to (26), we will have $d_1 = 0$ and $d_2 = 0$. Thus we get that $\lim_{r_0 \to 0} d' = 0$. And since we can choose $r_0$ to be arbitrarily small by reducing $\bar{\delta}_{\max}$, we can in turn make $d'$ arbitrarily small. Assume now that (21) holds for some $\delta_{\max} = \bar{\delta}_{\max}$. Then we can replace the constant $\lambda$ in (21) with a function $\lambda(\delta_{\max})$ such that (21) still holds for every $\delta_{\max} \leq \bar{\delta}_{\max}$, and furthermore, $\lim_{\delta_{\max}\searrow 0} \lambda(\delta_{\max}) = 0$. Looking at (22), it is easy to see that we can upper bound $\|\tilde{\boldsymbol{x}}_d\|_{[\Delta,\infty]}$ with $\bar{\gamma}_1(\delta_{\max}) + \bar{\gamma}_e(\bar{x}_{\max}; \delta_{\max})$ where

$$\bar{\gamma}_1(\delta_{\max}) \doteq \frac{1}{1 - \lambda(\delta_{\max})} \max_{\mu \in [0,\mu_{\max}]} \tilde{\beta}_e(x'_{\max}, 0; \mu),$$

and the remaining element on the left-hand side of (22) is represented by $\bar{\gamma}_e$. Thus (24) and the second limit result in (25) follows. Because $\lim_{\delta_{\max}\searrow 0} \lambda(\delta_{\max}) = 0$, the first limit result in (25) also follows. □

Another corollary of (Sharon and Liberzon, 2010, Theorem 4) is as follows:

*Corollary 9.* Assume that (10) and (24) holds $\forall \left|\tilde{\boldsymbol{x}}_d(\Delta)\right| \leq x'_{\max}$, $\forall \left\|\boldsymbol{x}_d\right\|_{[\Delta,\infty]} \leq \bar{x}_{\max}$, $\forall \mu_{k(\Delta)} \leq \mu_{\max}$, $\forall \delta_{\max} < \bar{\delta}_{\max}$ for some $x'_{\max}$, $\bar{x}_{\max}$, $\mu_{\max}$ and $\bar{\delta}_{\max}$. Set $r'_1 = \bar{x}_{\max}$. Assume also that for some $r'_0 > 0$, $\alpha > 0$ and $\lambda < 1$, $\forall r \in [r'_0, r'_1]$:

$$\gamma_x\left((1+\alpha)\,\bar{\gamma}_e\left(r;\delta_{\max}\right)\right) < \lambda r, \tag{27}$$

and

$$\begin{aligned}&\frac{1}{1-\lambda}\beta_x\left(\left|\boldsymbol{x}_d(\Delta)\right|,0\right) + \\ &\frac{1}{1-\lambda}\gamma_x\left(\left(1+\frac{1}{\alpha}\right)\bar{\gamma}_1\left(\delta_{\max}\right)\right) \qquad\qquad < r'_1, \end{aligned}\tag{28}$$

$$\begin{aligned}&\frac{1}{1-\lambda}\bar{\gamma}_1\left(\delta_{\max}\right) + \\ &\frac{1}{1-\lambda}\bar{\gamma}_e\left(\left(1+\frac{1}{\alpha}\right)\left(\beta_x\left(\left|\boldsymbol{x}_d(\Delta)\right|,0\right)\right);\delta_{\max}\right) < \gamma_x^{-1}\left(r'_1\right). \end{aligned}\tag{29}$$

Then $\left\|\boldsymbol{x}_d\right\|_{[\Delta,\infty]} < r'_1$ and $\left\|\tilde{\boldsymbol{x}}_d\right\|_{[\Delta,\infty]} < \gamma_x^{-1}\left(r'_1\right)$.

We remark that having (29) imply $\left\|\tilde{\boldsymbol{x}}_d\right\|_{[\Delta,\infty]} < \gamma_x^{-1}\left(r'_1\right)$ given (27) relies on the linearity of $\gamma_x$ which was established in Lemma 3.

We are now ready to prove Theorem 1

**Proof.** Assume $x'_{\max}$, $\mu_{\max}$ are given. Choose $r'_1$ such that

$$r'_1 > \beta_x\left(x'_{\max},0\right) + \gamma_x\left(\sup_{\mu\in[0,\mu_{\max}]}\tilde{\beta}_e\left(x'_{\max},0;\mu\right)\right). \tag{30}$$

We can now find $\bar{\delta}_{\max} > 0$ for which (23) holds with $\bar{x}_{\max} = r'_1$ and $\bar{\gamma}_1$, $\bar{\gamma}_e$ (due to (30) and (25)) such that

$$\begin{aligned}\beta_x\left(x'_{\max},0\right) + \gamma_x\left(\bar{\gamma}_1\right) &< r'_1 \\ \gamma_x\left(\bar{\gamma}_1 + \gamma_e\left(\beta_x\left(x'_{\max},0\right);\delta_{\max}\right)\right) &< r'_1.\end{aligned}$$

Taking a smaller $\bar{\delta}_{\max}$ if necessary, we can now also satisfy the assumptions of Corollary 9. This establishes that if $\boldsymbol{x}_d(\Delta) \leq x'_{max}$, $\tilde{\boldsymbol{x}}_d(\Delta) \leq x'_{max}$, $\mu_{k(\Delta)} \leq \mu_{\max}$, and $\delta_{\max} \leq \bar{\delta}_{\max}$ then $\forall t \geq t_0 \geq \Delta$ both (10) and (23) holds, as well as $\left\|\boldsymbol{x}_d\right\|_{[\Delta,\infty]} < r'_1$, and $\left\|\tilde{\boldsymbol{x}}_d\right\|_{[\Delta,\infty]} < \gamma_x^{-1}\left(r'_1\right)$.

Taking an even smaller $\bar{\delta}_{\max}$ if necessary, we can make the small-gain condition between (10) and (23),

$$\gamma_x\left(\gamma_e\left(\delta_{\max} r\right)\right) < r \qquad \forall r \in \left[r'_0, r'_1\right],$$

hold $\forall \delta_{\max} < \bar{\delta}_{\max}$, so that we can apply the local Small-Gain Theorem (Corollary 7) one more time and arrive at $\forall t \geq t_0 \geq \Delta$:

$$\left|\begin{pmatrix}\boldsymbol{x}_d(t)\\ \tilde{\boldsymbol{x}}_d(t)\end{pmatrix}\right| \leq \beta'\left(\left|\begin{pmatrix}\boldsymbol{x}_d(t_0)\\ \tilde{\boldsymbol{x}}_d(t_0)\end{pmatrix}\right|, t-t_0\right) + d \tag{31}$$

where $\beta' \in \mathcal{KL}$. The last term above, $d$, is nonzero due to $d' > 0$ and $r'_0 > 0$ when $\delta_{\max} > 0$. However, we can make both $d'$ and $r'_0$ arbitrarily small, and therefore also $d$, by taking a sufficiently small $\delta_{\max} > 0$. Thus we can replace $d$ with $\gamma\left(\delta_{\max}\right) \in \mathcal{K}$.

We now bound the evolvement of $\boldsymbol{x}$ and $\hat{\boldsymbol{x}}$ from $t = 0$ to $t = \Delta$. Initially $\hat{\boldsymbol{x}} = 0$ and at the first sampling by our quantizer $\left|\hat{\boldsymbol{x}}\left(\delta_0\right)\right| < 2\left|\boldsymbol{x}(0)\right|$, leading to $\left\|\hat{\boldsymbol{x}}\right\|_{[0,T_s+\delta_1]} \leq e^{(T_s+\delta_{\max})\|A+BK\|}2\left|\boldsymbol{x}(0)\right| \doteq \rho_1\left|\boldsymbol{x}(0)\right|$. Thus

$$\begin{aligned}\left\|\boldsymbol{x}\right\|_{[0,T_s+\delta_1]} &\leq e^{(T_s+\delta_{\max})\|A\|}\left|\boldsymbol{x}(0)\right| + \\ &\left(T_s+\delta_{\max}\right)e^{(T_s+\delta_{\max})\|A\|}\left\|BK\right\|\rho_1\left|\boldsymbol{x}(0)\right| \doteq \rho_2\left|\boldsymbol{x}(0)\right|.\end{aligned}$$

Then $\left|\tilde{\boldsymbol{x}}^-\left(T_s+\delta_1\right)\right| \leq \left(\rho_1+\rho_2\right)\left|\boldsymbol{x}(0)\right|$. Our quantizer has the property that $\left|\tilde{\boldsymbol{x}}\left(T_s+\delta_1\right)\right| \leq \left|\tilde{\boldsymbol{x}}^-\left(T_s+\delta_1\right)\right|$, so that $\left|\hat{\boldsymbol{x}}\left(T_s+\delta_1\right)\right| \leq \left(\rho_1+2\rho_2\right)\left|\boldsymbol{x}(0)\right|$. Repeating these arguments, we can derive the bound $\left|\boldsymbol{x}_d(\Delta)\right| \leq \rho\left|\boldsymbol{x}(0)\right|$ and $\left|\tilde{\boldsymbol{x}}_d(\Delta)\right| \leq \rho\left|\boldsymbol{x}(0)\right|$ for some $\rho > 0$.

Noting that $k(\Delta) = 2$, we can also bound $\mu_{k(\Delta)} \leq s\Omega_{out}^2$. To complete the proof, find $\bar{\delta}_{\max}$ such that (31) holds for $x'_{\max} = \rho x_{\max}$ and $\mu_{\max} = s\Omega_{out}^2$, and set

$$\beta(\nu;t) \doteq \beta'\left(\rho\nu, \max\{0, t-\Delta\}\right). \qquad \square$$

## 6. CONCLUSION

In this paper we showed that the “zoom out”/“zoom in” mechanism that we developed in Sharon and Liberzon (2010) maintains its stability property for sufficiently small delays. While we proved the existence of a non-trivial delay for which the system is still stable, we are yet to provide a constructive method to verify whether the system is stable for a given delay. We also plan to extend these results to non-linear systems, along the lines of (Sharon and Liberzon, 2010, Section VI), and to show that the stability to external delays we proved in our earlier work still holds in the presence of delays.

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